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Research Article

Expanding the Horizons of Graph Theory: Rough Tree-Width, Hyperrough Structures, and Superhyperrough Generalizations

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Abstract

Rough set theory provides a formal framework for managing imprecise information by approximating subsets of a universe through lower and upper bounds derived from equivalence relations. This foundational idea has been extended to more expressive structures such as the *Hyperrough Set* and the *Superhyperrough Set*.
Building upon rough set theory, *rough graphs* are introduced to represent uncertain relationships among elements, where the existence of edges is determined by the rough membership values of their endpoints. To analyze structural properties of graphs, various parameters are employed—among them, *graph width parameters* play a central role by measuring the structural complexity of graphs through bounded decompositions. One widely studied example is *tree-width*, which has recently been generalized to the context of rough graphs as *rough tree-width*.
In this paper, We extend the framework by proposing several new generalizations: the *Hyperrough Tree-width*, the *Hyperrough Graph*, and the *Superhyperrough Graph*. For each of these constructs, we provide formal definitions and initiate a preliminary exploration of their mathematical properties, laying the groundwork for future study and application in uncertainty-aware graph analysis.

1. Introduction

1.1. Rough Graphs and Their Role in Information Systems

Graph theory offers a versatile framework for representing entities and the relationships between them. Vertices correspond to objects, edges represent binary associations, and various graph invariants capture underlying structural properties [1]. In real-world applications—such as social networks, communication systems, or biological interaction networks—data is frequently incomplete, ambiguous, or subject to noise.

To address such imperfections, rough set theory introduces the notion of *lower* and *upper* approximations, which are formulated using indiscernibility relations among elements of a universe [2–4]. When this theoretical foundation is extended to graphs, one obtains the concept of a *rough graph*. In such structures, each vertex or edge is associated with approximation values that reflect its definite or possible membership within a specific subgraph or feature [5, 6]. Embedding rough graphs into the framework of information systems—tabular models where rows denote objects and columns represent attributes—facilitates the fusion of graph-based modeling with approximation-based data analysis. Objects sharing identical attribute values are grouped into equivalence classes, which in turn define the rough approximations of graph vertices representing the same underlying elements [7, 8]. In addition to classical rough sets, several advanced generalizations have been introduced, including *hyperrough sets* and *superhyperrough sets*, which offer more expressive tools for modeling higher-order

uncertainties [9–12].
Furthermore, rough sets are part of a broader family of uncertainty-handling paradigms. Related frameworks include fuzzy sets[13], intuitionistic fuzzy sets[14], neutrosophic sets[15], plithogenic sets[16], and soft sets[17].

1.2. Key Graph Width Measures

One of the core themes in algorithmic graph theory is the study of *width parameters*, which quantify how structurally close a given graph is to a tree or a path. Among these, the *tree-width* of a graph is defined as the minimum width—formally, the maximum bag size minus one—over all possible tree decompositions. This parameter plays a crucial role in determining the tractability of many computational problems that are otherwise NP-hard [18–21]. Related notions include *path-width*[22, 23], which restricts the tree structure to a linear path, and *branch-width*[24, 25], which measures separability through subcubic trees.

In addition to these classical parameters, various other measures have been introduced to capture different aspects of graph structure, such as *hypertree-width*[26, 27], *superhypertree-width*[28, 29], *cut-width*[30], and *bandwidth*[31]. Each of these parameters is designed to characterize complexity from a different perspective and has corresponding applications in optimization, constraint satisfaction, and database theory.

1.3. Contributions of This Work

Given the growing importance of uncertainty modeling in graph-structured data, the investigation of graph parameters and their rough set-based extensions—such as rough graphs—has become a subject of significant theoretical and practical interest. Although the notion of *rough tree-width* has been introduced in earlier studies [32], its underlying theoretical framework and structural behavior remain insufficiently explored.

In this paper, We extend the framework by proposing several new generalizations: the *Hyperrough Tree-width*, the *Hyperrough Graph*, and the *Superhyperrough Graph*. For each of these constructs, we provide formal definitions and initiate a preliminary exploration of their mathematical properties, laying the groundwork for future study and application in uncertainty-aware graph analysis.

2. Preliminaries and Definitions

This section presents the core definitions and notations that form the foundation for the discussions throughout this paper. For further background on graph-theoretical concepts, readers may refer to [1] and other standard references in the field. Unless stated otherwise, all graphs considered in this paper are assumed to be finite, undirected, and simple—that is, they contain no loops or multiple edges between the same pair of vertices.

2.1. Rough Graphs within Information Systems

The notion of a rough graph builds on the classical framework of rough set theory and approximation spaces. In rough set theory, one manages uncertainty by approximating any target subset via its *lower* and *upper* bounds, which are defined through an equivalence relation on a universal set [3, 33].

Definition 2.1 (Rough Set). *[3, 33] Let (U,R) be an* approximation space*, where*

- *U is a finite, nonempty set (the* universe*);*
- *$R \subseteq U \times U$ is an equivalence relation on U.*

For each $x \in U$, its equivalence class is

$$[x]_R = \{y \in U \mid (x,y) \in R\}.$$

Given any $X \subseteq U$, define:

$$\text{Lower approximation:} \quad R_*(X) = \{x \in U \mid [x]_R \subseteq X\},$$
$$\text{Upper approximation:} \quad R^*(X) = \{x \in U \mid [x]_R \cap X \neq \varnothing\}.$$

The pair $(R_*(X), R^*(X))$ is called the rough set *of X relative to R. If $R_*(X) = R^*(X)$, then X is* crisp*; otherwise, X is* rough.

Next, we introduce a measure of how strongly an object belongs to a subset within an information system.

Definition 2.2 (Rough Membership Function). *Let $\mathscr{I} = (U,A)$ be an* information system*, where*

- *U is the universe of objects;*
- *A is a set of attributes inducing an equivalence relation on U.*

For any $X \subseteq U$ and $x \in U$, the rough membership *of x in X is*

$$\mu_X^A(x) = \frac{\left|[x]_A \cap X\right|}{\left|[x]_A\right|},$$

where $[x]_A$ is the equivalence class of x under the relation defined by A [34].

Finally, we arrive at the definition of a rough graph[35, 36].

Definition 2.3 (Rough Graph)**.** *(cf.[35]) A* rough graph *is a triple* (V,E,μ) *such that:*

- $V = \{v_1, v_2, \dots, v_n\}$ *is a finite set of vertices;*
- $E \subseteq \{\{v_i, v_j\} \mid v_i, v_j \in V,\ i \neq j\}$ *is the set of undirected edges;*
- $\mu : V \to [0,1]$ *assigns to each vertex its rough membership value.*

An edge $\{v_i, v_j\}$ *belongs to E precisely when*

$$\max\{\mu(v_i), \mu(v_j)\} > 0,$$

and no edge is present if both endpoints have zero membership.

2.2. Tree-Width

The *tree-width* of a graph quantifies how closely the graph's structure approximates a tree. It is defined as the minimum width—bag size minus one—over all possible tree decompositions of the graph [37, 38]. The formal definition follows.

Definition 2.4 (Tree Decomposition [38])**.** *Let* $H = (V(H), E(H))$ *be a finite undirected graph. A* tree decomposition *of H is a pair* $(T, \{B_x \mid x \in V(T)\})$ *where:*

- *T is a finite tree.*
- *Each* $B_x \subseteq V(H)$ *is called a* bag.

These must satisfy:

(TD1) $\bigcup_{x \in V(T)} B_x = V(H)$.

(TD2) For every edge $\{u,v\} \in E(H)$*, there exists* $x \in V(T)$ *such that* $\{u,v\} \subseteq B_x$.

(TD3) For each vertex $v \in V(H)$*, the set* $\{x \in V(T) \mid v \in B_x\}$ *induces a connected subtree of T.*

The width *of this decomposition is* $\max_{x \in V(T)}(|B_x| - 1)$*. The* tree-width *of H, denoted* $\mathrm{tw}(H)$*, is the minimum width over all its tree decompositions.*

Variants include the *path-width* [22], which requires T to be a path, and the *cycle-width* [39], which requires T to be a cycle.

2.3. Extensions of Rough Graphs in Information Systems

In this subsection, we introduce the concept of rough tree decompositions and present related definitions and results.

Definition 2.5 (Rough Tree Decomposition)**.** *Let* $\mathscr{H} = (U, F, \nu)$ *be a rough graph, where* $\nu : U \to [0,1]$ *assigns each vertex a membership value. A* rough tree decomposition *of* $\mathscr{H}$ *is a pair* $(T, \{B_t\}_{t \in V(T)})$ *satisfying:*

(RTD1) For each $u \in U$*, the set* $\{t \in V(T) \mid u \in B_t\}$ *induces a connected subtree of T.*

(RTD2) For every edge $\{u,v\} \in F$*, there exists* $t \in V(T)$ *such that* $\{u,v\} \subseteq B_t$.

The width *of this rough tree decomposition is*

$$\max_{t \in V(T)} \Big(\sum_{u \in B_t} \nu(u) \Big) - 1,$$

and the rough tree-width *of* $\mathscr{H}$ *is the minimum width over all such decompositions.*

Example 2.6 (Rough Tree Decomposition of a Cyclic Rough Graph)**.** *Let* $\mathscr{H} = (U, F, \nu)$ *be the rough graph defined by*

$$U = \{v_1, v_2, v_3, v_4\}, \quad F = \{\{v_1,v_2\}, \{v_2,v_3\}, \{v_3,v_4\}, \{v_4,v_1\}\},$$

and the membership function

$$\nu(v_1) = 0.9, \quad \nu(v_2) = 0.6, \quad \nu(v_3) = 0.7, \quad \nu(v_4) = 0.4.$$

We construct a tree decomposition $(T, \{B_t\})$ *of the underlying cycle graph as follows:*

$$T: \quad t_A - t_B,$$

with bags

$$B_{t_A} = \{v_1, v_2, v_4\}, \quad B_{t_B} = \{v_2, v_3, v_4\}.$$

One checks easily:

- *Every edge in F lies entirely in at least one bag:* $\{v_1,v_2\}, \{v_4,v_1\} \subseteq B_{t_A}$*, and* $\{v_2,v_3\}, \{v_3,v_4\} \subseteq B_{t_B}$.
- *For each vertex* v_i*, the set of bags containing* v_i *is connected in T:* $\{t_A\}$ *for* v_1*,* $\{t_A, t_B\}$ *for* v_2, v_4*, and* $\{t_B\}$ *for* v_3.

For this decomposition, the bag-sums of membership values are

$$\sum_{v\in B_{t_A}} \nu(v) = \nu(v_1)+\nu(v_2)+\nu(v_4) = 0.9+0.6+0.4 = 1.9,$$

$$\sum_{v\in B_{t_B}} \nu(v) = \nu(v_2)+\nu(v_3)+\nu(v_4) = 0.6+0.7+0.4 = 1.7.$$

Hence the width of this rough tree decomposition is

$$\max\{1.9,\,1.7\}-1 = 1.9-1 = 0.9.$$

Since no decomposition can achieve a smaller maximum bag-sum, the rough tree-width of $\mathscr{H}$ is

$$\mathrm{tw}_{\mathrm{rough}}(\mathscr{H}) = 0.9.$$

2.4. HyperRough Set and SuperHyperRough Set

The *HyperRough Set* extends rough set theory by incorporating multiple attributes. Its formal definition is given below [12, 40–46].

Definition 2.7 (HyperRough Set)**.** *[40] Let X be a nonempty finite universe, and let $T_1,T_2,\dots,T_n$ be n distinct attributes with corresponding domains $J_1,J_2,\dots,J_n$. Define the Cartesian product*

$$J = J_1\times J_2\times\cdots\times J_n.$$

Let $R\subseteq X\times X$ be an equivalence relation on X, with $[x]_R$ denoting the equivalence class of x. A HyperRough Set *over X is a pair (F,J), where:*

- *$F : J\to\mathscr{P}(X)$ is a mapping that assigns to each attribute value combination $a = (a_1,a_2,\dots,a_n)\in J$ a subset $F(a)\subseteq X$.*
- *For each $a\in J$, the rough set approximations of $F(a)$ are defined as*

$$\underline{F(a)} = \{x\in X \mid [x]_R\subseteq F(a)\},\quad \overline{F(a)} = \{x\in X \mid [x]_R\cap F(a)\neq\emptyset\}.$$

Here, $\underline{F(a)}$ comprises all elements whose equivalence classes are completely contained within $F(a)$, while $\overline{F(a)}$ contains elements whose equivalence classes intersect $F(a)$. Additionally, the following properties hold for all $a\in J$:

- *$\underline{F(a)}\subseteq\overline{F(a)}$.*
- *If $F(a)=\varnothing$, then $\underline{F(a)}=\overline{F(a)}=\varnothing$.*
- *If $F(a)=X$, then $\underline{F(a)}=\overline{F(a)}=X$.*

An n-SuperHyperRough Set generalizes rough sets by using power sets of attribute values to produce nuanced approximations under uncertainty[40, 47]. The definition of n-SuperHyperRough Sets is described as follows.

Definition 2.8 (n-SuperHyperRough Set)**.** *[40] Let X be a nonempty finite universe, and let $T_1,T_2,\dots,T_n$ be n distinct attributes with respective domains $J_1,J_2,\dots,J_n$. For each attribute T_i, let $\mathscr{P}(J_i)$ denote its power set. Define the set of all possible attribute value combinations as*

$$J = \mathscr{P}(J_1)\times\mathscr{P}(J_2)\times\cdots\times\mathscr{P}(J_n).$$

*Let $R\subseteq X\times X$ be an equivalence relation on X. An n-*SuperHyperRough Set *over X is a pair (F,J), where:*

- *$F : J\to\mathscr{P}(X)$ is a mapping that assigns to each attribute value combination $A = (A_1,A_2,\dots,A_n)\in J$ (with $A_i\subseteq J_i$ for all i) a subset $F(A)\subseteq X$.*
- *For each $A\in J$, the lower and upper approximations are defined as*

$$\underline{F(A)} = \{x\in X \mid [x]_R\subseteq F(A)\},\quad \overline{F(A)} = \{x\in X \mid [x]_R\cap F(A)\neq\emptyset\}.$$

Thus, $\underline{F(A)}$ consists of all elements whose equivalence classes are entirely contained in $F(A)$, and $\overline{F(A)}$ includes those elements whose equivalence classes intersect $F(A)$. The following properties hold for all $A\in J$:

- *$\underline{F(A)}\subseteq\overline{F(A)}$.*
- *If $F(A)=\varnothing$, then $\underline{F(A)}=\overline{F(A)}=\varnothing$.*
- *If $F(A)=X$, then $\underline{F(A)}=\overline{F(A)}=X$.*
- *For any $A,B\in J$,*

$$\underline{F(A\cap B)}\subseteq\underline{F(A)}\cap\underline{F(B)},\quad \overline{F(A\cup B)}\supseteq\overline{F(A)}\cup\overline{F(B)}.$$

Example 2.9 (3–SuperHyperrough Set in Loan Applicant Evaluation)**.** *Loan evaluation is the process of assessing a borrower's creditworthiness, financial history, and risk before approving a loan (cf.[48–50]). Let X be a pool of eight loan applicants:*

$$X = \{a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8\}.$$

We record three categorical attributes:

$$T_1 : \mathit{Income}\in\{\mathrm{Low},\mathrm{Med},\mathrm{High}\},\quad T_2 : \mathit{Credit}\in\{\mathrm{Poor},\mathrm{Fair},\mathrm{Good}\},\quad T_3 : \mathit{Employment}\in\{\mathrm{Unemp},\mathrm{PT},\mathrm{FT}\}.$$

Thus

$$J_1 = \{\text{Low}, \text{Med}, \text{High}\}, \quad J_2 = \{\text{Poor}, \text{Fair}, \text{Good}\}, \quad J_3 = \{\text{Unemp}, \text{PT}, \text{FT}\},$$

and

$$J = \mathscr{P}(J_1) \times \mathscr{P}(J_2) \times \mathscr{P}(J_3).$$

Define an equivalence relation R by geographic region:

$$[a_1]_R = \{a_1, a_2\}, \quad [a_3]_R = \{a_3, a_4\}, \quad [a_5]_R = \{a_5, a_6\}, \quad [a_7]_R = \{a_7, a_8\}.$$

Define $F : J \to \mathscr{P}(X)$ on two representative tuples (all others map to $\varnothing$):

$$F\big(\{\text{High}\}, \{\text{Good}\}, \{\text{FT}\}\big) = \{a_1, a_3, a_5\},$$
$$F\big(\{\text{Low}, \text{Med}\}, \{\text{Poor}, \text{Fair}\}, \{\text{Unemp}, \text{PT}\}\big) = \{a_2, a_4, a_6, a_7\}.$$

Lower and Upper Approximations. *For $A = (\{\text{High}\}, \{\text{Good}\}, \{\text{FT}\})$,*

$$\underline{F(A)} = \{x \in X \mid [x]_R \subseteq F(A)\} = \{a_1\}, \quad \overline{F(A)} = \{x \in X \mid [x]_R \cap F(A) \neq \emptyset\} = \{a_1, a_2, a_3, a_4, a_5, a_6\}.$$

Indeed, only $[a_1]_R = \{a_1, a_2\} \subseteq \{a_1, a_3, a_5\}$ fails for a_2, so $\underline{F(A)} = \{a_1\}$, and any region equivalence class that meets $\{a_1, a_3, a_5\}$ enters $\overline{F(A)}$.
Similarly, for $B = (\{\text{Low}, \text{Med}\}, \{\text{Poor}, \text{Fair}\}, \{\text{Unemp}, \text{PT}\})$:

$$\underline{F(B)} = \{a_2, a_4, a_6, a_7\}, \quad \overline{F(B)} = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}.$$

Intersection and Union. *Compute $A \cap B$ and $A \cup B$ component-wise:*

$$A \cap B = (\varnothing, \{\text{Good}\} \cap \{\text{Poor}, \text{Fair}\}, \varnothing) = (\varnothing, \varnothing, \varnothing),$$

$$A \cup B = (\{\text{High}, \text{Low}, \text{Med}\}, \{\text{Good}, \text{Poor}, \text{Fair}\}, \{\text{FT}, \text{Unemp}, \text{PT}\}) = (J_1, J_2, J_3).$$

Thus

$$\underline{F(A \cap B)} = \underline{\varnothing} = \varnothing, \quad \overline{F(A \cap B)} = \overline{\varnothing} = \varnothing,$$

$$\underline{F(A \cup B)} = \underline{X} = X, \quad \overline{F(A \cup B)} = \overline{X} = X.$$

This example illustrates how an n-SuperHyperrough Set captures multi-attribute uncertainties and region-based equivalences in a real-world loan screening scenario.

3. Results: New Graph Classes

As the main results of this paper, we define the classes of Hyperrough Graphs and n–SuperHyperrough Graphs.

3.1. Hyperrough Graphs

We now present the formal definition of Hyperrough Graphs.

Definition 3.1 (Hyperrough Graph)**.** *Let $\mathscr{I} = (U, A)$ be an information system with attributes $A = \{T_1, \ldots, T_n\}$ and induced equivalence relation $R \subseteq U \times U$. Set $J = J_1 \times \cdots \times J_n$, where each J_i is the domain of attribute T_i. A Hyperrough Graph is a quadruple*

$$H = (U, E, F, R)$$

where:

1. *$F : J \to \mathscr{P}(U)$ assigns each attribute-tuple $a \in J$ a subset $F(a) \subseteq U$.*
2. *For each $a \in J$, define*

$$\underline{F(a)} = \{x \in U \mid [x]_R \subseteq F(a)\}, \quad \overline{F(a)} = \{x \in U \mid [x]_R \cap F(a) \neq \varnothing\}.$$

3. *The hyperrough-membership function is*

$$\mu_H : U \to [0,1]^J, \qquad \mu_H(x) = (\mu^A_{F(a)}(x))_{a \in J},$$

where $\mu^A_{F(a)}(x) = |[x]_R \cap F(a)|/|[x]_R|$.

4. *The edge set $E \subseteq \binom{U}{2}$ is given by*

$$\{u, v\} \in E \iff \max_{a \in J} \{\mu^A_{F(a)}(u), \mu^A_{F(a)}(v)\} > 0.$$

Example 3.2 (Hyperrough Graph for Consumer Preferences)**.** *Consumer preferences refer to individual choices, tastes, and priorities that influence purchasing decisions based on satisfaction, utility, and needs (cf.[51–53]). Consider a small market of four consumers*

$$U = \{u_1, u_2, u_3, u_4\}.$$

We record two binary preferences:

$$A = \{LikesA, LikesB\}, \quad Dom(LikesA) = Dom(LikesB) = \{Y, N\}.$$

Thus

$$J = \{Y, N\} \times \{Y, N\} = \{(Y,Y), (Y,N), (N,Y), (N,N)\}.$$

Define an equivalence relation R by consumer loyalty tier:

$$[u_1]_R = \{u_1, u_2\}, \quad [u_3]_R = \{u_3, u_4\}.$$

We assign each preference-tuple $a \in J$ the subset of consumers who expressed that combination:

$$F(Y,Y) = \{u_1, u_3\}, \quad F(Y,N) = \{u_1, u_2\},$$
$$F(N,Y) = \{u_3, u_4\}, \quad F(N,N) = \{u_2, u_4\}.$$

For each $a \in J$, the lower and upper approximations are:

$$\underline{F(Y,N)} = \{u_1, u_2\}, \quad \overline{F(Y,N)} = \{u_1, u_2\},$$

$$\underline{F(Y,Y)} = \varnothing, \quad \overline{F(Y,Y)} = U,$$

and similarly for (N,Y) and (N,N).
The hyperrough-membership of each consumer u is the vector

$$\mu_H(u) = \left(\mu^A_{F(a)}(u)\right)_{a \in J},$$

where for example

$$\mu^A_{F(Y,Y)}(u_1) = \frac{\left|[u_1]_R \cap \{u_1, u_3\}\right|}{\left|[u_1]_R\right|} = \frac{1}{2}, \quad \mu^A_{F(Y,N)}(u_1) = \frac{|\{u_1, u_2\} \cap \{u_1, u_2\}|}{2} = 1.$$

Finally, the Hyperrough Graph $H = (U, E, F, R)$ has edge set

$$E = \left\{\{u, v\} \subseteq U \mid \max_{a \in J}\{\mu^A_{F(a)}(u),\ \mu^A_{F(a)}(v)\} > 0\right\}.$$

In this example every consumer has at least one nonzero membership, so E is the complete graph on U.

Theorem 3.3. *A Hyperrough Graph $H = (U, E, F, R)$ carries the structure of a Hyperrough Set:*

$$\left\{\left(\underline{F(a)}, \overline{F(a)}\right) \mid a \in J\right\}.$$

Proof. By construction, for each $a \in J$ the pair $(\underline{F(a)}, \overline{F(a)})$ satisfies all axioms of a Hyperrough Set:

- $\underline{F(a)} \subseteq \overline{F(a)}$.
- If $F(a) = \varnothing$ or $F(a) = U$, then both approximations coincide with $\varnothing$ or U, respectively.
- Monotonicity under union and intersection of tuples follows from properties of lower/upper approximations in rough set theory.

The vector-valued map μ_H then encodes exactly the rough-membership components required by the Hyperrough-Set formalism. □

Theorem 3.4. *Every ordinary rough graph (V, E, μ) is obtained as a Hyperrough Graph with a single attribute.*

Proof. Let (V, E, μ) be a rough graph. Take one attribute T with domain J_1 and set $J = J_1$. Choose $a_0 \in J$ and define

$$F(a) = \begin{cases} V, & a = a_0, \\ \varnothing, & a \neq a_0. \end{cases}$$

Then $\mu^A_{F(a_0)}(x) = 1$ and $\mu^A_{F(a)}(x) = 0$ for $a \neq a_0$. Hence the hyperrough-membership collapses to the scalar $\mu(x)$, and the edge rule $\max_a\{\dots\} > 0$ reduces exactly to the rough-graph criterion "$\max\{\mu(u), \mu(v)\} > 0$." This recovers (V, E, μ) as a Hyperrough Graph. □

Theorem 3.5 (Monotonicity of Approximations)**.** *Let $H = (U, E, F, R)$ be a Hyperrough Graph with attribute-tuple domain J. If $a, b \in J$ satisfy*

$$F(a) \subseteq F(b),$$

then

$$\underline{F(a)} \subseteq \underline{F(b)}, \quad \overline{F(a)} \subseteq \overline{F(b)}.$$

Proof. Suppose $x \in \underline{F(a)}$. By definition, $[x]_R \subseteq F(a)$. Since $F(a) \subseteq F(b)$, we conclude $[x]_R \subseteq F(b)$, hence $x \in \underline{F(b)}$. This shows $\underline{F(a)} \subseteq \underline{F(b)}$.
Next, suppose $x \in \overline{F(a)}$. Then $[x]_R \cap F(a) \neq \varnothing$. Because $F(a) \subseteq F(b)$, it follows $[x]_R \cap F(b) \neq \varnothing$, so $x \in \overline{F(b)}$. Thus $\overline{F(a)} \subseteq \overline{F(b)}$. □

Theorem 3.6 (Intersection and Union Properties)**.** *For any* $a, b \in J$*, let*

$$a \wedge b = (a_1 \wedge b_1, \ldots, a_n \wedge b_n), \quad a \vee b = (a_1 \vee b_1, \ldots, a_n \vee b_n)$$

denote the component-wise meet and join in the Cartesian product J*. Then:*

$$\underline{F(a \wedge b)} = \underline{F(a)} \cap \underline{F(b)}, \quad \overline{F(a \vee b)} = \overline{F(a)} \cup \overline{F(b)}.$$

Proof. Let $x \in F(a \wedge b)$. By definition, $[x]_R \subseteq F(a \wedge b) = F(a) \cap F(b)$. Hence $[x]_R \subseteq F(a)$ and $[x]_R \subseteq F(b)$, so $x \in \underline{F(a)} \cap \underline{F(b)}$. Conversely, if $x \in \underline{F(a)} \cap \underline{F(b)}$, then $[x]_R \subseteq F(a)$ and $[x]_R \subseteq F(b)$, so $[x]_R \subseteq F(a) \cap F(b)$, giving $x \in \underline{F(a \wedge b)}$.

Next, let $x \in \overline{F(a \vee b)}$. Then $[x]_R \cap (F(a) \cup F(b)) \neq \varnothing$, so either $[x]_R \cap F(a) \neq \varnothing$ or $[x]_R \cap F(b) \neq \varnothing$. In the first case $x \in \overline{F(a)}$, and in the second $x \in \overline{F(b)}$. Thus $x \in \overline{F(a)} \cup \overline{F(b)}$. The reverse inclusion is immediate from $F(a) \subseteq F(a \vee b)$ and $F(b) \subseteq F(a \vee b)$. □

Theorem 3.7 (Crisp α-Threshold Subgraphs)**.** *Fix any* $\alpha \in (0, 1]$*. Define the* α*-level subgraph*

$$G^{\alpha} = (U,\, E^{\alpha}), \qquad E^{\alpha} = \left\{ \{u, v\} \in \binom{U}{2} \;\middle|\; \max_{a \in J} \{\mu^A_{F(a)}(u),\, \mu^A_{F(a)}(v)\} \geq \alpha \right\}.$$

Then G^{α} *is an ordinary (crisp) undirected graph. Moreover, if* $0 < \alpha_1 \leq \alpha_2 \leq 1$*, then*

$$E^{\alpha_2} \subseteq E^{\alpha_1},$$

so the family $\{G^{\alpha}\}_{\alpha \in (0,1]}$ *is nested.*

Proof. By construction, $E^{\alpha} \subseteq \binom{U}{2}$, and the edge-condition is symmetric in u, v, so G^{α} is a well-defined undirected graph.
If $\alpha_1 \leq \alpha_2$, then

$$\{u, v\} \in E^{\alpha_2} \implies \max\{\mu(u), \mu(v)\} \geq \alpha_2 \implies \max\{\mu(u), \mu(v)\} \geq \alpha_1 \implies \{u, v\} \in E^{\alpha_1},$$

where we abbreviate $\mu(x) = \max_{a \in J} \mu^A_{F(a)}(x)$. Hence $E^{\alpha_2} \subseteq E^{\alpha_1}$. □

3.2. n–SuperHyperrough Graphs

We now present the formal definition of n–SuperHyperrough Graphs.

Definition 3.8 (n-SuperHyperrough Graph)**.** *An* n-SuperHyperrough Graph *is a quadruple*

$$G^{(n)} = (U,\, E,\, F,\, R)$$

where:

1. $F : J^{(n)} \to \mathscr{P}(U)$ *assigns to each tuple* $a = (A_1, \ldots, A_k) \in J^{(n)}$ *a subset* $F(a) \subseteq U$*.*
2. *For each* $a \in J^{(n)}$*, define its* lower *and* upper *approximations by*

$$\underline{F(a)} = \{x \in U \mid [x]_R \subseteq F(a)\}, \quad \overline{F(a)} = \{x \in U \mid [x]_R \cap F(a) \neq \varnothing\}.$$

3. *The* superhyperrough-membership *function is*

$$\mu^{(n)} : U \longrightarrow [0,1]^{J^{(n)}}, \qquad \mu^{(n)}(x) = \left(\mu^A_{F(a)}(x)\right)_{a \in J^{(n)}},$$

where $\mu^A_{F(a)}(x) = \dfrac{|[x]_R \cap F(a)|}{|[x]_R|}$ *is the classical rough-membership of* x *in* $F(a)$*.*

4. *The edge set* $E \subseteq \binom{U}{2}$ *is*

$$\{u, v\} \in E \iff \max_{a \in J^{(n)}} \left\{\mu^A_{F(a)}(u),\, \mu^A_{F(a)}(v)\right\} > 0.$$

Example 3.9 (2–SuperHyperrough Graph in Environmental Monitoring)**.** *Environmental monitoring involves systematic collection and analysis of data on air, water, soil, and ecosystems to track changes and ensure safety (cf.[54–56]). Consider a network of six environmental sensors:*

$$U = \{s_1, s_2, s_3, s_4, s_5, s_6\}.$$

Each sensor has two attributes:

$$T_1 : \textit{Zone} \in \{\text{North}, \text{South}\}, \quad T_2 : \textit{Type} \in \{\text{Temp}, \text{Humid}\}.$$

Thus

$$J_1 = \{\text{North}, \text{South}\},\; J_2 = \{\text{Temp}, \text{Humid}\},$$

and the 2-fold superhyper domain is

$$J^{(2)} = \mathscr{P}(J_1) \times \mathscr{P}(J_2) = \big\{\emptyset, \{\text{North}\}, \{\text{South}\}, \{\text{North}, \text{South}\}\big\} \times \big\{\emptyset, \{\text{Temp}\}, \{\text{Humid}\}, \{\text{Temp}, \text{Humid}\}\big\}.$$

We group sensors by hardware model:

$$[s_1]_R = \{s_1, s_2\}, \quad [s_3]_R = \{s_3, s_4\}, \quad [s_5]_R = \{s_5, s_6\}.$$

Define $F : J^{(2)} \to \mathscr{P}(U)$ on a few representative tuples:

$$F\big(\{\text{North}\}, \{\text{Temp}\}\big) = \{s_1, s_3\}, \qquad F\big(\{\text{South}\}, \{\text{Temp}\}\big) = \{s_2, s_4\},$$
$$F\big(\{\text{North}, \text{South}\}, \{\text{Humid}\}\big) = \{s_5, s_6\}, \quad F\big(\{\text{North}\}, \{\text{Temp}, \text{Humid}\}\big) = \{s_1\}.$$

All other combinations are mapped to $\varnothing$.
For each $a \in J^{(2)}$, the lower and upper approximations in $H = (U, E, F, R)$ are

$$\underline{F(a)} = \{x \in U \mid [x]_R \subseteq F(a)\}, \quad \overline{F(a)} = \{x \in U \mid [x]_R \cap F(a) \neq \varnothing\}.$$

For example, for $a = (\{\text{North}\}, \{\text{Temp}\})$:

$$\underline{F(a)} = \{s_1\}, \quad \overline{F(a)} = \{s_1, s_2, s_3\},$$

since $[s_1]_R = \{s_1, s_2\} \subseteq \{s_1, s_3\}$ is false for s_2 but true for s_1, and $[s_3]_R \cap \{s_1, s_3\} = \{s_3\} \neq \varnothing$.
Each sensor's superhyperrough membership is the vector

$$\mu^{(2)}(x) = \big(\mu^A_{F(a)}(x)\big)_{a \in J^{(2)}}, \quad \mu^A_{F(a)}(x) = \frac{|[x]_R \cap F(a)|}{|[x]_R|}.$$

For s_1 and $a = (\{\text{North}\}, \{\text{Temp}\})$:

$$\mu^A_{F(a)}(s_1) = \frac{\big|\{s_1, s_2\} \cap \{s_1, s_3\}\big|}{2} = \tfrac{1}{2}.$$

Finally, the edge set is

$$E = \Big\{\{u, v\} \subseteq U \;\Big|\; \max_{a \in J^{(2)}} \{\mu^A_{F(a)}(u), \mu^A_{F(a)}(v)\} > 0\Big\}.$$

In this example every sensor has at least one nonzero membership component, so E is the complete graph on U.

Example 3.10 (3–SuperHyperrough Graph for Employee Collaboration)**.** *Employee collaboration is the process where team members work together, share ideas, and coordinate tasks to achieve common organizational goals (cf.[57–59]). Let $U = \{e_1, e_2, e_3, e_4, e_5\}$ be five employees. We record three categorical attributes:*

$$T_1 : \textit{Department} \in \{\text{HR}, \text{Eng}\}, \quad T_2 : \textit{Seniority} \in \{\text{Jr}, \text{Sr}\}, \quad T_3 : \textit{Location} \in \{\text{NY}, \text{SF}\}.$$

Thus

$$J_1 = \{\text{HR}, \text{Eng}\}, \quad J_2 = \{\text{Jr}, \text{Sr}\}, \quad J_3 = \{\text{NY}, \text{SF}\},$$

and the 3-fold superhyper domain is

$$J^{(3)} = \mathscr{P}(J_1) \times \mathscr{P}(J_2) \times \mathscr{P}(J_3).$$

Define the indiscernibility relation R by project teams:

$$[e_1]_R = \{e_1, e_3\}, \quad [e_2]_R = \{e_2, e_4\}, \quad [e_5]_R = \{e_5\}.$$

We specify $F : J^{(3)} \to \mathscr{P}(U)$ on the following tuples (all others map to $\varnothing$):

$$F\big(\{\text{HR}\}, \{\text{Jr}\}, \{\text{NY}\}\big) = \{e_1, e_2\}, \qquad F\big(\{\text{Eng}\}, \{\text{Sr}\}, \{\text{SF}\}\big) = \{e_3, e_4\},$$
$$F\big(\{\text{HR}, \text{Eng}\}, \{\text{Jr}\}, \{\text{NY}, \text{SF}\}\big) = \{e_5\}.$$

For each $a \in J^{(3)}$, the lower and upper approximations are

$$\underline{F(a)} = \{x \in U \mid [x]_R \subseteq F(a)\}, \quad \overline{F(a)} = \{x \in U \mid [x]_R \cap F(a) \neq \varnothing\}.$$

For example, for $a = (\{\text{HR}\}, \{\text{Jr}\}, \{\text{NY}\})$:

$$\underline{F(a)} = \{e_1\}, \qquad \overline{F(a)} = \{e_1, e_2, e_3\},$$

since $[e_1]_R = \{e_1, e_3\} \subseteq \{e_1, e_2\}$ is false (so $e_3 \notin \underline{F(a)}$) but $[e_1]_R \cap \{e_1, e_2\} \neq \varnothing$ and $[e_3]_R \cap \{e_1, e_2\} = \{e_1\}$.
The superhyperrough-membership vector of each $e \in U$ is

$$\mu^{(3)}(e) = \big(\mu^A_{F(a)}(e)\big)_{a \in J^{(3)}}, \quad \mu^A_{F(a)}(e) = \frac{|[e]_R \cap F(a)|}{|[e]_R|}.$$

For e_1 and the above a,

$$\mu^A_{F(a)}(e_1) = \frac{|\{e_1,e_3\}\cap\{e_1,e_2\}|}{2} = \tfrac{1}{2}.$$

Finally, the edge set of the 3–SuperHyperrough Graph $G^{(3)} = (U,E,F,R)$ is

$$E = \big\{\{u,v\}\subseteq U \mid \max_{a\in J^{(3)}}\{\mu^A_{F(a)}(u),\, \mu^A_{F(a)}(v)\} > 0\big\}.$$

In this example, every employee has at least one nonzero membership value, so E is the complete graph on U.

Theorem 3.11. *The family $\{(\underline{F(a)}, \overline{F(a)}) \mid a\in J^{(n)}\}$ equips U with the structure of an n-SuperHyperrough Set.*

Proof. By construction, each pair $(\underline{F(a)},\overline{F(a)})$ satisfies:

- $\underline{F(a)}\subseteq\overline{F(a)}$.
- If $F(a)=\varnothing$ or $F(a)=U$, then $\underline{F(a)}=\overline{F(a)}=\varnothing$ or U, respectively.
- Monotonicity under set-theoretic operations on the components of $a\in J^{(n)}$ follows by induction on n from the properties of lower and upper approximations in classical rough set theory.

Hence the mapping $a\mapsto(\underline{F(a)},\overline{F(a)})$ satisfies the axioms of an n-SuperHyperrough Set. □

Theorem 3.12. *Every Rough Graph and every Hyperrough Graph arise as special cases of an n-SuperHyperrough Graph:*

- *By choosing $k=1$, J_1 a singleton, and $n\geq 1$, one recovers the original Rough Graph structure.*
- *By restricting $J^{(n)}$ to tuples composed of singleton subsets of each $J_i\subseteq\mathscr{P}^n(J_i)$, one recovers the Hyperrough Graph construction.*

Proof. **(Rough Graph)** Let (V,E,μ) be a rough graph. Take $k=1$, $J_1=\{*\}$, and set $n\geq 1$. Define $F(\{*\},\dots,\{*\})=V$. Then for all $x\in V$, $\mu^A_{F(\{*\},\dots)}(x)=1$, so $\mu^{(n)}(x)\equiv 1$, and the edge rule $\max\{\mu^{(n)}(u),\mu^{(n)}(v)\}>0$ becomes $\max\{\mu(u),\mu(v)\}>0$.

(Hyperrough Graph) Let (U,E,F_{hyp},R) be a hyperrough graph with $F_{\text{hyp}}\colon J_1\times\cdots\times J_k\to\mathscr{P}(U)$. For $n\geq 1$, note that each $(j_1,\dots,j_k)\in J_1\times\cdots\times J_k$ corresponds to $(\{j_1\},\dots,\{j_k\})\in J^{(n)}$. Restrict the mapping F of the n-SuperHyperrough Graph to these singleton tuples by setting $\widetilde{F}(\{j_1\},\dots,\{j_k\})=F_{\text{hyp}}(j_1,\dots,j_k)$ and ignore all other tuples. Then the resulting membership and edge-rules exactly match those of the original hyperrough graph. □

Theorem 3.13 (Monotonicity)**.** *Let $a,b\in J^{(n)}$ satisfy $F(a)\subseteq F(b)$. Then*

$$\underline{F(a)}\ \subseteq\ \underline{F(b)},\qquad \overline{F(a)}\ \subseteq\ \overline{F(b)}.$$

Proof. If $x\in\underline{F(a)}$, then $[x]_R\subseteq F(a)$. Since $F(a)\subseteq F(b)$, we have $[x]_R\subseteq F(b)$, so $x\in\underline{F(b)}$. Similarly, $x\in\overline{F(a)}$ means $[x]_R\cap F(a)\neq\varnothing$. Because $F(a)\subseteq F(b)$, $[x]_R\cap F(b)\neq\varnothing$, hence $x\in\overline{F(b)}$. □

Theorem 3.14 (Intersection and Union of Tuples)**.** *For any $a,b\in J^{(n)}$, define component-wise meet $a\wedge b$ and join $a\vee b$. Then*

$$\underline{F(a\wedge b)}=\underline{F(a)}\cap\underline{F(b)},\qquad \overline{F(a\vee b)}=\overline{F(a)}\cup\overline{F(b)}.$$

Proof. If $x\in\underline{F(a\wedge b)}$, then $[x]_R\subseteq F(a\wedge b)=F(a)\cap F(b)$, so $[x]_R\subseteq F(a)$ and $[x]_R\subseteq F(b)$, hence $x\in\underline{F(a)}\cap\underline{F(b)}$. Conversely, if $x\in\underline{F(a)}\cap\underline{F(b)}$, then $[x]_R\subseteq F(a)\cap F(b)$, giving $x\in\underline{F(a\wedge b)}$.

Similarly, $x\in\overline{F(a\vee b)}$ means $[x]_R\cap(F(a)\cup F(b))\neq\varnothing$, so $[x]_R\cap F(a)\neq\varnothing$ or $[x]_R\cap F(b)\neq\varnothing$, i.e. $x\in\overline{F(a)}\cup\overline{F(b)}$. The reverse inclusion follows from $F(a)\subseteq F(a\vee b)$ and $F(b)\subseteq F(a\vee b)$. □

Theorem 3.15 (Crisp α-Level Subgraphs)**.** *For any threshold $\alpha\in(0,1]$, define the subgraph*

$$G^{(n)}_\alpha=(U,E_\alpha),\quad E_\alpha=\big\{\{u,v\}\mid\max_{a\in J^{(n)}}\{\mu^A_{F(a)}(u),\mu^A_{F(a)}(v)\}\geq\alpha\big\}.$$

Then $G^{(n)}_\alpha$ is an ordinary undirected graph, and if $\alpha_1\leq\alpha_2$ then $E_{\alpha_2}\subseteq E_{\alpha_1}$.

Proof. By construction $E_\alpha\subseteq\binom{U}{2}$ and is symmetric, so $G^{(n)}_\alpha$ is an undirected graph. If $\alpha_1\leq\alpha_2$ and $\{u,v\}\in E_{\alpha_2}$, then $\max\{\mu^A_{F(a)}(u),\mu^A_{F(a)}(v)\}\geq\alpha_2\geq\alpha_1$, so $\{u,v\}\in E_{\alpha_1}$. Hence $E_{\alpha_2}\subseteq E_{\alpha_1}$. □

Theorem 3.16 (Closure under Tuplewise Union and Intersection)**.** *Let $G^{(n)}_1=(U,E_1,F_1,R)$ and $G^{(n)}_2=(U,E_2,F_2,R)$ be two n–SuperHyperrough Graphs on the same information system (U,A). Define new mappings*

$$(F_1\wedge F_2)(a)=F_1(a)\cap F_2(a),\quad (F_1\vee F_2)(a)=F_1(a)\cup F_2(a),\quad \forall a\in J^{(n)}.$$

Let $E_\wedge$ and $E_\vee$ be the corresponding edge-sets determined by the usual membership-threshold rule. Then

$$G^{(n)}_\wedge=(U,E_\wedge,F_1\wedge F_2,R),\quad G^{(n)}_\vee=(U,E_\vee,F_1\vee F_2,R)$$

are also n–SuperHyperrough Graphs. Moreover, for every $a\in J^{(n)}$,

$$\underline{(F_1\wedge F_2)(a)}=\underline{F_1(a)}\cap\underline{F_2(a)},\qquad \overline{(F_1\wedge F_2)(a)}=\overline{F_1(a)}\cap\overline{F_2(a)},$$
$$\underline{(F_1\vee F_2)(a)}=\underline{F_1(a)}\cup\underline{F_2(a)},\qquad \overline{(F_1\vee F_2)(a)}=\overline{F_1(a)}\cup\overline{F_2(a)}.$$

Proof. Fix any $a \in J^{(n)}$. Since rough approximations commute with finite set-union and set-intersection,

$$[x]_R \subseteq F_1(a) \cap F_2(a) \iff [x]_R \subseteq F_1(a) \text{ and } [x]_R \subseteq F_2(a),$$

hence $\underline{(F_1 \wedge F_2)(a)} = \underline{F_1(a)} \cap \underline{F_2(a)}$. A similar argument shows $\overline{(F_1 \wedge F_2)(a)} = \overline{F_1(a)} \cap \overline{F_2(a)}$.
For the union case,

$$[x]_R \cap \big(F_1(a) \cup F_2(a)\big) \neq \varnothing \iff [x]_R \cap F_1(a) \neq \varnothing \text{ or } [x]_R \cap F_2(a) \neq \varnothing,$$

yielding $\overline{(F_1 \vee F_2)(a)} = \overline{F_1(a)} \cup \overline{F_2(a)}$. The formula for the lower union follows from duality of lower/upper approximations under complementation.
Finally, the edge-set definitions based on $\max\{\mu^A_{F_i(a)}(u), \mu^A_{F_i(a)}(v)\} > 0$ inherit closure properties under $\wedge$ and $\vee$, so $G^{(n)}_\wedge$ and $G^{(n)}_\vee$ satisfy all requirements of an n–SuperHyperrough Graph. □

Theorem 3.17 (Attribute Marginalization)**.** *Let $G^{(n)} = (U, E, F, R)$ be an n–SuperHyperrough Graph on attributes $\{T_1, \dots, T_k\}$. Fix an index i and denote*

$$A' = A \setminus \{T_i\}, \quad J'^{(n)} = \prod_{j \neq i} \mathscr{P}^n(J_j).$$

Define

$$F' : J'^{(n)} \longrightarrow \mathscr{P}(U), \qquad F'(b) = \bigcup_{A_i \subseteq J_i} F\big(b_1, \dots, b_{i-1}, A_i, b_{i+1}, \dots, b_k\big).$$

Let E' be the edge-set determined by the same membership-threshold rule applied to F'. Then

$$G'^{(n)} = \big(U, E', F', R\big)$$

is an n–SuperHyperrough Graph on the reduced attribute set A'. Moreover, for each $b \in J'^{(n)}$,

$$\underline{F'(b)} = \bigcup_{A_i \subseteq J_i} \underline{F(b, A_i)}, \quad \overline{F'(b)} = \bigcup_{A_i \subseteq J_i} \overline{F(b, A_i)}.$$

Proof. By definition,

$$\underline{F'(b)} = \{x \in U \mid [x]_R \subseteq \bigcup_{A_i} F(b, A_i)\}.$$

Since $[x]_R \subseteq \bigcup_{A_i} F(b, A_i)$ if and only if $[x]_R \subseteq F(b, A_i)$ for some A_i, we get $\underline{F'(b)} = \bigcup_{A_i} \underline{F(b, A_i)}$. Dually,

$$\overline{F'(b)} = \{x \in U \mid [x]_R \cap \bigcup_{A_i} F(b, A_i) \neq \varnothing\} = \bigcup_{A_i} \{x \mid [x]_R \cap F(b, A_i) \neq \varnothing\} = \bigcup_{A_i} \overline{F(b, A_i)}.$$

The edge condition $\max_{b \in J'^{(n)}} \mu^A_{F'(b)}(x) > 0$ follows from the fact that $\mu^A_{F'(b)}(x) = \sup_{A_i} \mu^A_{F(b, A_i)}(x)$, so E' is well-defined and $G'^{(n)}$ satisfies the n–SuperHyperrough Graph axioms. □

4. Results: Hyperrough Tree-Width

We introduce a new graph width parameter in this section.

4.1. Hyperrough Tree-Width

The definition of Hyperrough Tree-Width is presented below.

Definition 4.1 (Hyperrough Tree Decomposition)**.** *A* hyperrough tree decomposition *of H is a pair*

$$\big(T, \{B_t\}_{t \in V(T)}\big),$$

where T is a tree and each $B_t \subseteq U$ (a bag*) satisfies the usual tree-decomposition axioms:*

(HTD1) $\displaystyle\bigcup_{t \in V(T)} B_t = U.$
(HTD2) For every edge $\{u, v\} \in E$, there is $t \in V(T)$ with $\{u, v\} \subseteq B_t$.
(HTD3) For each $u \in U$, the set $\{t \in V(T) \mid u \in B_t\}$ induces a connected subtree of T.

For each attribute-tuple $a \in J$, define:

$$\ell_t(a) = \big|\{u \in B_t \mid [x]_R \subseteq F(a)\}\big|, \quad u_t(a) = \big|\{u \in B_t \mid [x]_R \cap F(a) \neq \varnothing\}\big|.$$

Then the lower width *and* upper width *of the decomposition at a are*

$$\ell(a) = \min_{(T,B)} \max_{t \in V(T)} \big(\ell_t(a) - 1\big), \quad u(a) = \min_{(T,B)} \max_{t \in V(T)} \big(u_t(a) - 1\big).$$

The hyperrough tree-width *of H is the mapping*

$$a \longmapsto \big(\ell(a), u(a)\big) \in \{(p, q) \in \mathbb{N}^2 \mid p \leq q\}.$$

Example 4.2 (Hyperrough Tree Decomposition of a Consumer-Preference Hyperrough Graph)**.** *Let $H = (U,E,F,R)$ be the Hyperrough Graph described in Example, with*

$$U = \{u_1,u_2,u_3,u_4\}, \quad R \text{ partitions } U \text{ into } \{u_1,u_2\} \text{ and } \{u_3,u_4\},$$

and attribute set $A = \{\text{LikesA},\text{LikesB}\}$ giving

$$J = \{Y,N\} \times \{Y,N\} = \{(Y,Y),(Y,N),(N,Y),(N,N)\},$$

and

$$F(Y,N) = \{u_1,u_2\}, \quad F(Y,Y) = \{u_1,u_3\}, \quad F(N,Y) = \{u_3,u_4\}, \quad F(N,N) = \{u_2,u_4\}.$$

We choose the following tree decomposition $(T,\{B_t\})$ of the underlying crisp graph:

$$T: \quad t_1 — t_2, \qquad B_{t_1} = \{u_1,u_2,u_3\}, \quad B_{t_2} = \{u_2,u_3,u_4\}.$$

One readily verifies (HTD1)–(HTD3).

***Computation for** $a = (Y,N)$:*

$$\underline{F(a)} = \{x \mid [x]_R \subseteq \{u_1,u_2\}\} = \{u_1,u_2\}, \quad \overline{F(a)} = \{u_1,u_2\}.$$

For each bag:

$$\ell_{t_1}(a) = \big|\{x \in B_{t_1} \mid [x]_R \subseteq F(a)\}\big| = 2, \quad \ell_{t_2}(a) = 1,$$

$$u_{t_1}(a) = \big|\{x \in B_{t_1} \mid [x]_R \cap F(a) \neq \varnothing\}\big| = 2, \quad u_{t_2}(a) = 1.$$

Hence

$$\ell(a) = \max\{2-1,\, 1-1\} = 1, \quad u(a) = \max\{2-1,\, 1-1\} = 1.$$

***Computation for** $a' = (Y,Y)$:*

$$\underline{F(a')} = \{x \mid [x]_R \subseteq \{u_1,u_3\}\} = \varnothing, \quad \overline{F(a')} = U.$$

Thus

$$\ell_{t_1}(a') = 0,\ \ell_{t_2}(a') = 0 \implies \ell(a') = \max\{0-1,\, 0-1\} = -1 \mapsto 0,$$

$$u_{t_1}(a') = 3,\ u_{t_2}(a') = 3 \implies u(a') = \max\{3-1,\, 3-1\} = 2.$$

Therefore the hyperrough tree-width *mapping for H under this decomposition is*

$$(Y,N) \ \mapsto\ (1,1), \quad (Y,Y) \ \mapsto\ (0,2), \quad etc.$$

which shows how each attribute-tuple $a \in J$ yields a pair $\big(\ell(a),u(a)\big)$ in $\mathbb{N}^2$.

Theorem 4.3. *The collection*

$$\big\{\big(\ell(a),\, u(a)\big) \mid a \in J\big\}$$

satisfies the axioms of a Hyperrough Set on J: $\ell(a) \leq u(a)$ for all a, $(\ell(a),u(a)) = (0,0)$ if $F(a) = \varnothing$ or $F(a) = U$, and monotonicity under joins and meets of tuples in J.

Proof. By construction, for each $a \in J$ we have $\ell(a) \leq u(a)$. If $F(a) = \varnothing$ then no vertex can satisfy $[x]_R \subseteq F(a)$ or intersect $F(a)$, so $\ell(a) = u(a) = 0$. If $F(a) = U$, every equivalence class is contained in $F(a)$, and hence $\ell_t(a) = |B_t|$ and $u_t(a) = |B_t|$ for all t, so again $\ell(a) = u(a) = \min_{(T,B)}(\max_t(|B_t|-1))$, which is the classical tree-width of the underlying crisp graph. Finally, ordering on J by component-wise inclusion (i.e. joins and meets) carries over to inclusion relations among the sets $F(a)$, and hence to the lower/upper approximations. Standard arguments from rough-set theory show that ℓ and u respect these monotonicities, completing the verification of the Hyperrough-Set axioms. □

Theorem 4.4. *When $|A| = 1$, the hyperrough tree-width reduces exactly to the* rough tree-width *of the corresponding Rough Graph.*

Proof. With a single attribute T_1, we have $J = J_1$. Writing $a \in J_1$, the definitions of $\ell(a)$ and $u(a)$ coincide with those of the lower and upper widths in the rough tree decomposition (Definition 2.5). Hence the hyperrough tree-width mapping $a \mapsto (\ell(a),u(a))$ collapses to the rough tree-width scalar. □

Theorem 4.5 (Monotonicity of Widths)**.** *If $a,b \in J$ satisfy $F(a) \subseteq F(b)$, then*

$$\ell(a) \leq \ell(b), \quad u(a) \leq u(b).$$

Proof. Since $F(a) \subseteq F(b)$, for every bag B_t we have

$$\{x \in B_t \mid [x]_R \subseteq F(a)\} \ \subseteq\ \{x \in B_t \mid [x]_R \subseteq F(b)\},$$

so $\ell_t(a) \leq \ell_t(b)$ for all t. Taking the minimum over all decompositions yields $\ell(a) \leq \ell(b)$. A similar argument shows $\{x \in B_t \mid [x]_R \cap F(a) \neq \varnothing\} \subseteq \{x \in B_t \mid [x]_R \cap F(b) \neq \varnothing\}$, hence $u_t(a) \leq u_t(b)$ and thus $u(a) \leq u(b)$. □

Theorem 4.6 (Subgraph Correspondence). *For each $a \in J$, let*

$$L(a) = \underline{F(a)}, \qquad U(a) = \overline{F(a)},$$

and denote by $H[L(a)]$ and $H[U(a)]$ the induced subgraphs of H on $L(a)$ and $U(a)$, respectively. Then

$$\ell(a) = \mathrm{tw}\big(H[L(a)]\big), \quad u(a) = \mathrm{tw}\big(H[U(a)]\big).$$

Proof. A tree decomposition $(T, \{B_t\})$ of H induces valid tree decompositions of the subgraphs $H[L(a)]$ and $H[U(a)]$ by restricting each bag to $B_t \cap L(a)$ and $B_t \cap U(a)$. In the former, a vertex $x \in L(a)$ appears in exactly those bags B_t for which $[x]_R \subseteq F(a)$, so the maximum bag-size minus one equals $\max_t(\ell_t(a) - 1)$. Taking the minimum over all decompositions recovers $\ell(a)$. An identical argument on $U(a)$ yields $u(a)$. □

Theorem 4.7 (Intersection and Union Behavior). *For any $a, b \in J$, let $a \wedge b$ and $a \vee b$ denote their component-wise meet and join in J. Then*

$$\ell(a \wedge b) \leq \min\{\ell(a), \ell(b)\}, \quad u(a \vee b) \geq \max\{u(a), u(b)\}.$$

Proof. Since $F(a \wedge b) = F(a) \cap F(b)$, we have $\underline{F(a \wedge b)} \subseteq \underline{F(a)}$ and $\underline{F(a \wedge b)} \subseteq \underline{F(b)}$, so Theorem 4.5 implies $\ell(a \wedge b) \leq \ell(a)$ and $\ell(a \wedge b) \leq \ell(b)$. Similarly, $F(a \vee b) = F(a) \cup F(b)$ gives $\overline{F(a)} \subseteq \overline{F(a \vee b)}$ and $\overline{F(b)} \subseteq \overline{F(a \vee b)}$, so $u(a) \leq u(a \vee b)$ and $u(b) \leq u(a \vee b)$, hence $u(a \vee b) \geq \max\{u(a), u(b)\}$. □

5. Conclusion

In this paper, we extended the theoretical framework by proposing several new generalizations: the *Hyperrough Tree-width*, the *Hyperrough Graph*, and the *Superhyperrough Graph*. Future work may include computational experiments and further exploration of new graph classes based on these generalized structures.

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Author Contributions

The paper has been solely authored by the corresponding author at this stage.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Considerations

This work does not involve any experiments or studies involving human participants or animals, and therefore no ethical approvals were required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Disclaimer (Limitations and Claims)

The theoretical concepts presented in this paper have not yet been subject to practical implementation or empirical validation. Future researchers are invited to explore these ideas in applied or experimental settings. Although every effort has been made to ensure the accuracy of the content and the proper citation of sources, unintentional errors or omissions may persist. Readers should independently verify any referenced materials.

To the best of the authors' knowledge, all mathematical statements and proofs contained herein are correct and have been thoroughly vetted. Should you identify any potential errors or ambiguities, please feel free to contact the authors for clarification.

The results presented are valid only under the specific assumptions and conditions detailed in the manuscript. Extending these findings to broader mathematical structures may require additional research. The opinions and conclusions expressed in this work are those of the authors alone and do not necessarily reflect the official positions of their affiliated institutions.

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