Research Article

The basins of attraction for optimal fourth-eighth order methods

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Article Info

Abstract

We developed a completely new, eighth- and fourth-order optimal approach for solving nonlinear equations in this study. With three function evaluations, the new methods' convergence order is four; with four function evaluations, it is eight. Furthermore, according to the Kung-Traub hypothesis, it is optimal. In order to illustrate their dynamic behaviour on the complex plane, the basins of attraction are provided.

1. Introduction

The nonlinear equation $f(x) = 0$ is one of the most significant problems in engineering, scientific computing, and applied mathematics in general. The method that is most frequently used to solve nonlinear equations is Newton's iteration method. Newton's approach has been refined by other academics to achieve higher order convergence and more accurate findings, see example [1–7]. In addition, the efficiency index (EI) is a widely used technique to evaluate the effectiveness of various iterative approaches. The definition of this index is $p^{1/m}$, where m is the number of functional evaluations required at each iteration and p is the convergence order. If and only if the iterative method with m functional evaluations has an order of convergence equal to 2^{m-1}, according to the conjecture of [8]. The most effective iterative methods for varied convergence orders have been developed by numerous scholars. Typically, the composition methodology is used to construct an optimal method, along with a few interpolations and approximations to minimise the amount of functional evaluations needed at each iteration. Various optimal fourth order and eighth order iterative techniques were developed, see for example [1, 9–12]. Further, we studied the behaviour of iterative scheme in the complex plane. Furthermore, a number of researchers have applied these concepts to many iterative schemes [1–6, 13–15], which discussed the basin of attraction of a few well-known iterative schemes.

The rest of the paper is set up as follows. The proposed strategies have been developed and their convergence analysis is covered in Section 2. Section 3 uses basins of attraction to study the suggested methods in the complex plane. Section 4 provides concluding observations.

2. Construction of proposed methods

We will define an Iterative Function (I.F.) by $x_{n+1} = \psi(x)$. Using the additional information at $x, \phi_1(x), ..., \phi_i(x), i \geq 1$, let x_{n+1} be calculated. Nothing from the past is utilised. Consequently,

$$x_{n+1} = \psi(x, \phi_1(x), ..., \phi_i(x)). \tag{1}$$

A multipoint I.F. without memory is then defined as ψ.

The Newton-Raphson (also known as Newton-I.F.) ($2^{nd}NR$) is provided by

$$\psi_{2^{nd}NR}(x) = x - u(x), u(x) = \frac{f(x)}{f'(x)}. \tag{2}$$

With two function evaluations, the $2^{nd}NR$ I.F. is a one-point I.F. that meets the Kung-Traub conjecture for $d = 2$. Also, $EI_{2^{nd}NR} = 1.414$.

2.1. Proposed optimal fourth order I.F

In this way, we attempt to derive a new optimal fourth order I.F.

$$\psi_{4^{th}PM}(x) = \psi_{2^{nd}NR}(x) - H(\tau)\frac{f(\psi_{2^{nd}NR}(x))}{f'(x)}.$$

$$H(\tau) = H(1) + (\tau - 1) H'(1) + (\tau - 1)^2 H''(1) + ... \text{ and } \tau = 1 - \frac{f(\psi_{2^{nd}NR}(x))}{f(x)}. \tag{3}$$

In the following Theorem we discuss the choice of the parameter $|H''(1)|$ for which the proposed method (3) has the optimal fourth order convergence.

Assume that the function $f : D \subset \mathbb{R} \to \mathbb{R}$ has continuous derivatives and is suitably smooth. If x_0 is selected in a suitably small neighborhood of x^* and $f(x)$ has a simple root x^* in the open interval D, then the approach (3) has fourth order convergence, when

$$H(1) = 1, \ H'(1) = -2, \ |H''(1)| < \infty \tag{4}$$

The error equation is satisfied.

$$e_{n+1} = \left((5 - \frac{H''(1)}{2})c_2^3 - c_2c_3\right)e^4 + O(e^5). \tag{5}$$

$c_j = \frac{f^{(j)}(x^*)}{j! \, f'(x^*)}$, $j = 2, 3, 4, ...$ and $e = x - x^*$.

Proof. Let $\tilde{e} = \psi_{2^{nd}NR}(x) - x^*$, $\hat{e} = \psi_{4^{th}PM}(x) - x^*$. Extending $f(x)$ and $f'(x)$ Around x^* using Taylor's technique, we have

$$f(x) = f'(x^*)\left(e + c_2 e^2 + c_3 e^3 + c_4e^4 + c_5 e^5 + c_6e^6 + c_7e^7 + c_8e^8 + O(e^9)\right) \tag{6}$$

and

$$f'(x) = f'(x^*)\left(1 + 2c_2 e + 3c_3e^2 + 4c_4e^3 + 5c_5 e^4 + 6c_6e^5 + 7c_7e^6 + 8c_8e^7 + 9c_9 e^8 + O(e^9)\right) \tag{7}$$

Thus,

$$\begin{aligned}
\tilde{e} &= c_2e^2 + \left(2c_3 - 2c_2^2\right)e^3 + \left(-7c_2c_3 + 4c_2^3 + 3c_4\right)e^4 + \left(-8c_2^4\right. \\
&\quad \left. + 20 c_2^2c_3 - 6 c_3^2 - 10c_2c_4 + 4c_5\right) e^5 + \left(16c_2^5 - 52 c_2^3c_3 + 28c_2^2c_4 - 17 c_3c_4\right. \\
&\quad \left. + c_2(33c_3^2 - 13c_5) + 5c_6\right) e^6 - 2\left(16 c_2^6 - 64 c_2^4c_3 - 9 c_3^3 + 36 c_2^3c_4 + 6c_4^2 + 9c_2^2(7c_3^2\right. \\
&\quad \left. - 2c_5) + 11c_3c_5 + c_2(-46c_3c_4 + 8 c_6) - 3c_7\right) e^7 + \left(64c_2^7 - 304c_2^5c_3\right. \\
&\quad + 176c_2^4c_4 + 75c_3^2c_4 + c_2^3(408 c_3^2 - 92c_5) - 31c_4c_5 - 27c_3c_6 \\
&\quad \left. + c_2^2(-348 c_3c_4 + 44c_6) + c_2(-135 c_3^3 + 64 c_4^2 + 118 c_3c_5 - 19c_7) + 7c_8\right) e^8 + ...
\end{aligned} \tag{8}$$

Using Taylor's approach, we may expand $f(\psi_{2^{nd}NR}(x))$ about x^* and obtain

$$f(\psi_{2^{nd}NR}(x)) = f'(x^*)\left(\tilde{e} + c_2\tilde{e}^2 + c_3\tilde{e}^3 + c_4\tilde{e}^4 + O(\tilde{e}^5)\right) \tag{9}$$

We obtain by simplifying and substituting these equations (6)-(8) and (4) in the (3).

$$\psi_{4^{th}PM}(x) - x^* = \left((5 - \frac{H''(0)}{2})c_2^3 - c_2c_3\right)e^4 + O(e^5).$$

This shows that fourth-order convergence is achieved by the suggested classes of approaches. □

We are able to generate a new optimal fourth order method in (4) by selecting any random value for $H''(1)$. Selecting $H''(1) = 2$ yields new suggested approaches as follows.

$$\psi_{4^{th}PM}(x) = \psi_{2^{nd}NR}(x) - H(\tau)\frac{f(\psi_{2^{nd}NR}(x))}{f'(x)}.$$

$$H(\tau) = 1 - 2(\tau - 1) + (\tau - 1)^2 \text{ and } \tau = 1 - \frac{f(\psi_{2^{nd}NR}(x))}{f(x)}. \tag{10}$$

This method (10) has the following error equation $\psi_{4^{th}PM}(x) - x^* = \left(3c_2^3 - c_2c_3\right)e^4 + O(e^5)$. $EI_{4thPM} = 1.587$ is the efficiency of the method (15).

2.2. An eighth-order optimum technique

Next, we try the following method to obtain a new optimal eighth order I.F.

$$\psi_{8^{th}PM}(x) = \psi_{4^{th}PM}(x) - \frac{f(\psi_{4^{th}PM}(x))}{f'(\psi_{4^{th}PM}(x))}.$$

With five function evaluations, the aforementioned one exhibits eighth order convergence. However, this is not the best approach. In order to estimate $f'(\psi_{4^{th}PM}(x))$, we must minimise a function while maintaining the same convergence order. This polynomial is used to estimate the optimal.

$$q(t) = b_3(t-x)^3 + b_2(t-x)^2 + b_1(t-x) + b_0, \tag{11}$$

which fulfils

$$q'(x) = f'(x), \; q(x) = f(x), \; q(\psi_{2^{nd}NR}(x)) = f(\psi_{2^{nd}NR}(x)), \; q(\psi_{4^{th}PM}(x)) = f(\psi_{4^{th}PM}(x)).$$

When the aforementioned requirements are applied to (11), there are generated four linear equations: b_0, b_1, b_2, and b_3. $b_0 = f(x)$ and $b_1 = f'(x)$ follow from $q(x) = f(x), q'(x) = f'(x)$. b_2 and b_3 are found by solving these equations:

$$
\begin{aligned}
f(\psi_{2^{nd}NR}(x)) &= b_3(\psi_{2^{nd}NR}(x)-x)^3 + b_2(\psi_{2^{nd}NR}(x)-x)^2 + f'(x)(\psi_{2^{nd}NR}(x)-x) + f(x) \\
f(\psi_{4^{th}PM}(x)) &= b_3(\psi_{4^{th}PM}(x)-x)^3 + b_2(\psi_{4^{th}PM}(x)-x)^2 + f'(x)(\psi_{4^{th}PM}(x)-x) + f(x).
\end{aligned}
$$

Therefore, by using divided differences, the aforementioned equations become simpler to

$$f[\psi_{2^{nd}NR}(x),x,x] = b_2 + b_3(\psi_{2^{nd}NR}(x)-x) \tag{12}$$

$$f[\psi_{4^{th}PM}(x),x,x] = b_2 + b_3(\psi_{4^{th}PM}(x)-x) \tag{13}$$

Equations (12) and (13) can be solved to yield

$$
\begin{aligned}
b_2 &= \frac{f[\psi_{2^{nd}NR}(x),\,x,\,x](\psi_{4^{th}PM}(x)-x) - f[\psi_{4^{th}PM}(x),\,x,\,x](\psi_{2^{nd}NR}(x)-x)}{\psi_{4^{th}PM}(x) - \psi_{2^{nd}NR}(x)}, \\
b_3 &= \frac{f[\psi_{4^{th}PM}(x),\,x,\,x] - f[\psi_{2^{nd}NR}(x),\,x,\,x]}{\psi_{4^{th}PM}(x) - \psi_{2^{nd}NR}(x)}.
\end{aligned}
\tag{14}
$$

Furthermore, we have the estimation using eq. (14).

$$f'(\psi_{4^{th}PM}(x)) \approx q'(\psi_{4^{th}PM}(x)) = b_1 + 2\,b_2(\psi_{4^{th}PM}(x)-x) + 3\,b_3(\psi_{4^{th}PM}(x)-x)^2.$$

Lastly, we provide a fresh, eighth-order optimum technique as

$$\psi_{8^{th}PM}(x) = \psi_{4^{th}PM}(x) - \frac{f(\psi_{4^{th}PM}(x))}{f'(x) + 2b_2(\psi_{4^{th}PM}(x)-x) + 3b_3(\psi_{4^{th}PM}(x)-x)^2}. \tag{15}$$

$EI_{8thYM} = 1.682$ is the efficiency of the approach (15).
We use MATHEMATICA software to demonstrate the convergence analysis of the suggested *I.F.s* (15). Assume that the function $f : D \subset \mathbb{R} \to \mathbb{R}$ is sufficiently smooth and has derivatives that are continuous. When x_0 is selected within a suitably small neighbourhood of x^* and $f(x)$ has a simple root x^* in the open interval D, the (15) is of eighth order convergence and fulfils the error equation:

$$\psi_{8^{th}PM}(x) - x^* = c_2^2\left(4c_2^2 - c_3\right)\left(4c_2^3 - c_2c_3 + c_4\right)e^8 + O(e^9) \tag{16}$$

3. Basins of attraction

Analysing the rational function's dynamic behaviour in relation to an iterative process provides valuable insights into the method's stability and convergence. [14] and [16] provide fundamental definitions and dynamic notions of rational functions.
Applying our iterative methods, We pick a square with 256×256 points that is $\mathbb{R} \times \mathbb{R} = [-2,2] \times [-2,2]$. We start in every $z^{(0)}$ in the square. If, for a maximum of 100 iterations, the sequence generated by the iterative technique attempts a zero z_j^* of the polynomial with a tolerance $|f(z^{(k)})| < 1e-4$, we conclude that $z^{(0)}$ is in the basin of attraction of this zero. We label this point $z^{(0)}$ with colours if $|z^{(N)} - z_j^*| < 1e-4$. This is done if the iterative technique, it begins in $z^{(0)}$ and, in N iterations ($N \le 100$), reaches a zero. We determine that the starting point has diverged if $N > 50$, and we apply a dark blue colour. The following describes the basins of attraction for the Newton's method and a few higher order Newton-type methods for finding the complex roots of the polynomials $p_1(z) = z^3 - 1$ and $p_2(z) = z^5 - 1$.
Figure 1 displays the polynomiographs for the approaches to the polynomials $p_1(z)$ and $p_2(z)$ for the NR_2. The polynomiographs for the fourth order iterative approaches for the polynomial $p_1(z)$ are displayed in Figure 2. The polynomiographs for the ninth order iterative approaches for the polynomial $p_1(z)$ are displayed in Figure 3. The polynomiographs for the fourth order iterative approaches for the polynomial $p_2(z)$ are displayed in Figure 4. The polynomiographs for the ninth order iterative approaches for the polynomial $p_2(z)$ are displayed in Figure 5.
It is noted that the performance of the approaches NR_2, $4_{th}PM$, and $8_{th}PM$ is remarkable in the $p_1(z)$. In close proximity of the boundary points, the methods SB_4, KT_8, and LW_8, exhibit some chaotic behaviour. In this scenario, the approaches CM_4, SJ_4, $PNPD_8$, SA_8, and $CFGT_8$ are sensitive to the initial guess selection.
Also note that the approaches $4_{th}PM$ and $4_{th}PM$ exhibit some chaotic behaviour in the vicinity of the boundary points for $p_2(z)$. NR_2, SB_4, CM_4, and SJ_4 are the techniques KT_8, In this instance, the values of LW_8, $PNPD_8$, SA_8, and $CFGT_8$ are all sensitive to the initial guess made.

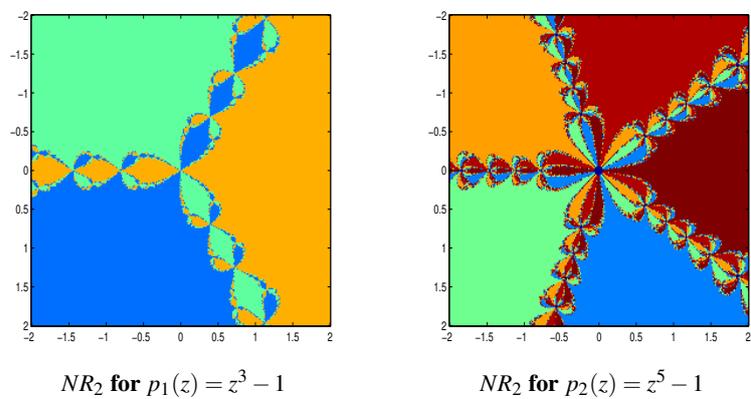NR_2 **for** $p_1(z) = z^3 - 1$ $\qquad$ NR_2 **for** $p_2(z) = z^5 - 1$

Figure 1: Newton's technique (NR_2) basins of attraction

SB_4 $\qquad\qquad$ CM_4

SJ_4 $\qquad\qquad$ $4_{th}PM$
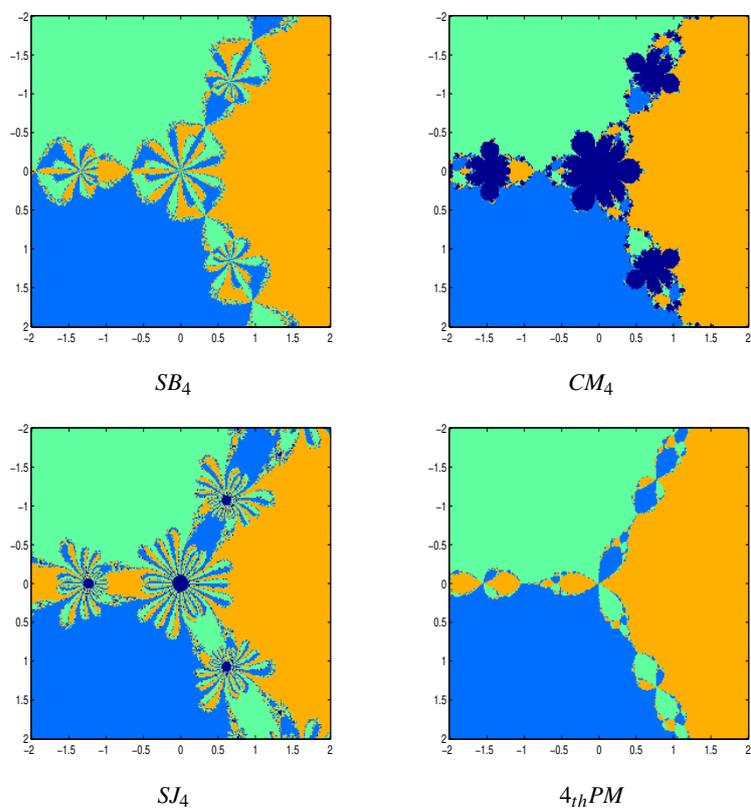
Figure 2: $p_1(z) = z^3 - 1$ basins of attraction

$PNPD_8$

SA_8

KT_8

$CFGT_8$

LW_8

$8_{th}PM$
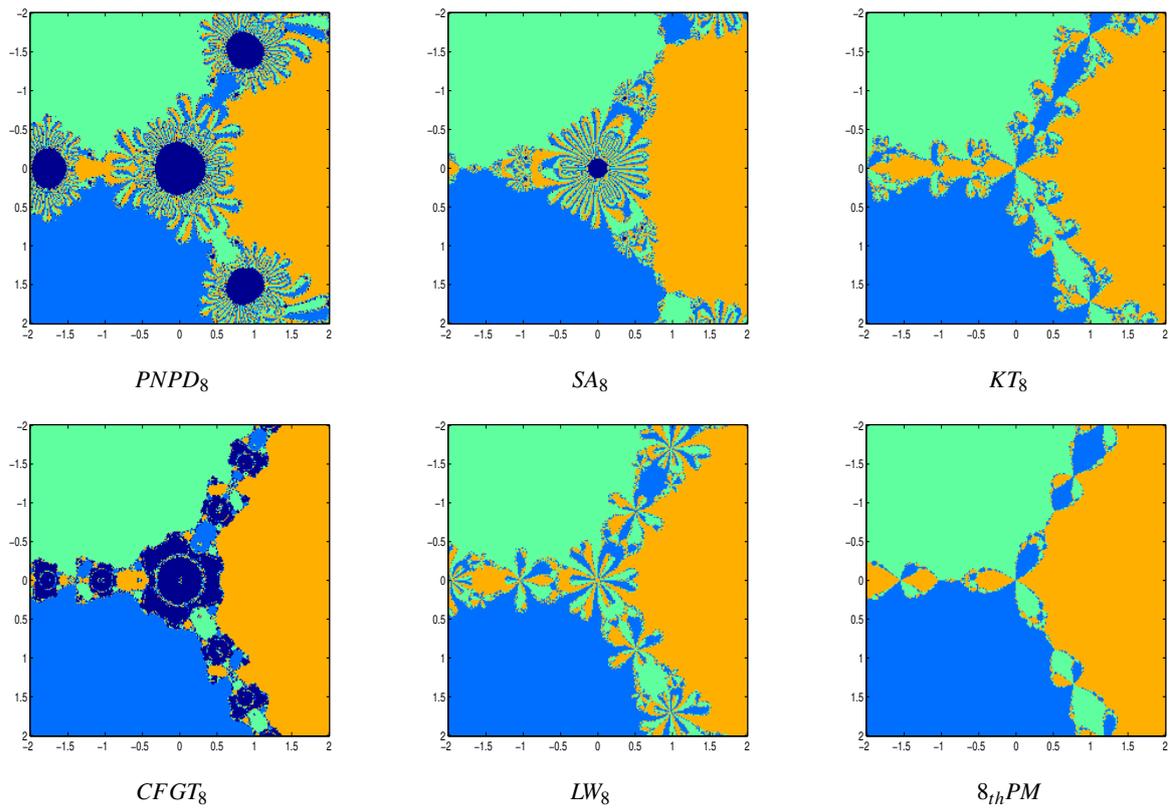
Figure 3: $p_1(z) = z^3 - 1$ basins of attraction

SB_4

CM_4

SJ_4

$4_{th}PM$
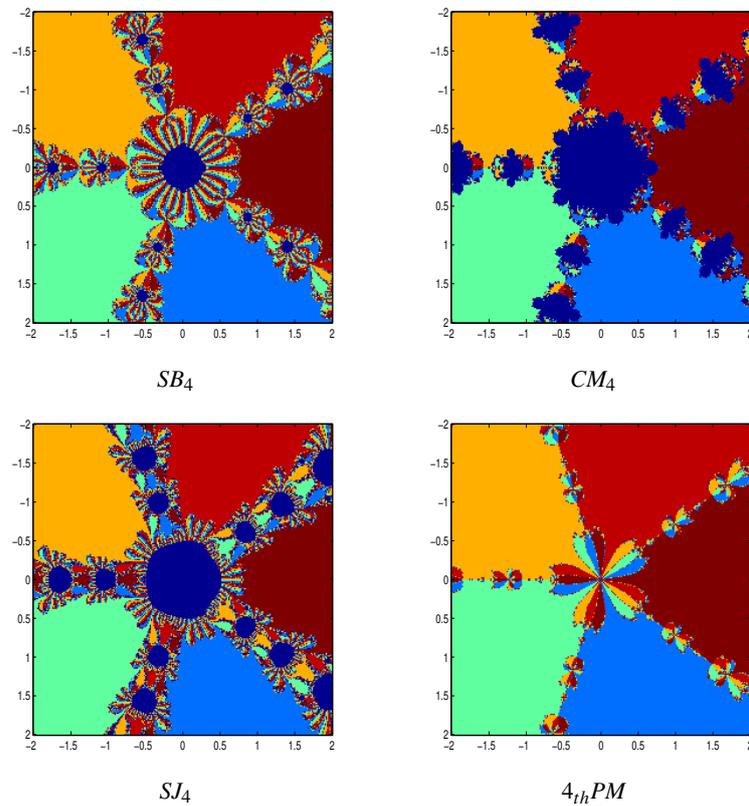
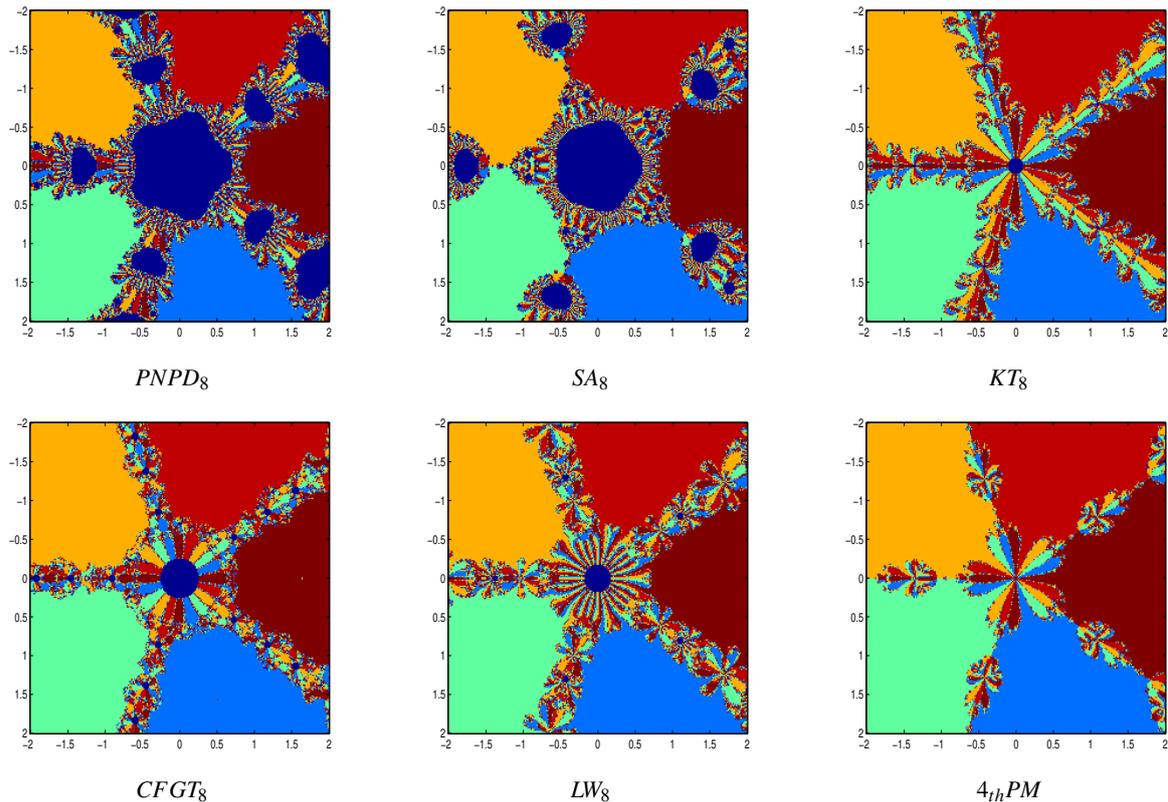**Figure 4:** $p_1(z) = z^5 - 1$ basins of attraction

Figure 5: $p_1(z) = z^5 - 1$ basins of attraction.

4. Concluding Remarks

In this work, we established a family of iterative algorithms for solving nonlinear equations that is optimal at the fourth and eighth orders, respectively. Three and four function evaluations are needed for the approach to obtain an order of convergence of four and eight, respectively. The Kung-Traub conjecture is met in the sense of convergence analysis and numerical examples. To demonstrate the superiority of the proposed methods, we have tested a few examples with both the proposed schemes and certain recognised schemes. By displaying their corresponding fractals, more research has been done on the complex plane to uncover the basins of attraction of such approaches for solving nonlinear equations.

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