

Research Article

Five trigonometric addition laws on semigroups

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Abstract

In this paper, we determine the complex-valued solutions of the following functional equations:

$$g(x\sigma(y)) = g(x)g(y) + f(x)f(y), \quad x, y \in S,$$

$$f(x\sigma(y)) = f(x)g(y) + f(y)g(x), \quad x, y \in S,$$

$$f(x\sigma(y)) = f(x)g(y) + f(y)g(x) - g(x)g(y), \quad x, y \in S,$$

$$f(x\sigma(y)) = f(x)g(y) + f(y)g(x) + \alpha g(x\sigma(y)), \quad x, y \in S,$$

$$f(x\sigma(y)) = f(x)g(y) - f(y)g(x) + \alpha g(x\sigma(y)), \quad x, y \in S,$$

where S is a semigroup, $\alpha \in \mathbb{C} \setminus \{0\}$ is a fixed constant, and $\sigma : S \rightarrow S$ is an involutive automorphism. We also provide some applications.

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1. Notations and terminology

In order to build our results, we recall the following notations and notions that will be used throughout the paper. A semigroup S is a set equipped with an associative binary operation. A multiplicative function on S is a function $\mu : S \rightarrow \mathbb{C}$ satisfying $\mu(xy) = \mu(x)\mu(y)$ for all $x, y \in S$. A function $a : S \rightarrow \mathbb{C}$ is additive if

$$a(xy) = a(x) + a(y) \quad \text{for all } x, y \in S.$$

A function $f : S \rightarrow \mathbb{C}$ is central if $f(xy) = f(yx)$ for all $x, y \in S$, and f is abelian if f is central and $f(xyz) = f(xzy)$ for all $x, y, z \in S$. Let S be a semigroup and $\sigma : S \rightarrow S$ an involutive automorphism. For a subset $T \subseteq S$, we define

$$T^2 := \{xy \mid x, y \in T\}.$$

If $\chi : S \rightarrow \mathbb{C}$ is a non-zero multiplicative function, we define the sets

$$I_\chi := \{x \in S \mid \chi(x) = 0\},$$

$$P_{\chi} := \{p \in I_{\chi} \setminus I_{\chi}^2 \mid up, pv, upv \in I_{\chi} \setminus I_{\chi}^2 \text{ for all } u, v \in S \setminus I_{\chi}\}.$$

For any function $f : S \rightarrow \mathbb{C}$, we define the functions

$$f^* = f \circ \sigma, \quad f^e = \frac{f + f^*}{2}, \quad f^o = \frac{f - f^*}{2}.$$

We say that f is even if $f^* = f$, and f is odd if $f^* = -f$.

For a topological semigroup S , let $C(S)$ denote the algebra of continuous functions from S into \mathbb{C} .

The following are respectively [1, Theorem 3.2], [2, Theorem 3.1], and [3, Proposition 3.2], but some formulas of the solutions are expressed in terms of a solution $\phi : S \rightarrow \mathbb{C}$ of (12).

Theorem 1.1. *The solutions $g, f : S \rightarrow \mathbb{C}$ of the functional equation*

$$g(xy) = g(x)g(y) - f(x)f(y), \quad x, y \in S,$$

are the following pairs:

- (1) $g = f = 0$.
- (2) $g = \frac{\delta^{-1}\chi_1 + \delta\chi_2}{\delta^{-1} + \delta}$ and $f = \frac{\chi_1 - \chi_2}{i(\delta^{-1} + \delta)}$, where $\chi, \chi_2 : S \rightarrow \mathbb{C}$ are two different multiplicative functions and $\delta \in \mathbb{C} \setminus \{0, i, -i\}$.
- (3) f is any non-zero function such that $f = 0$ on S^2 and $g = \pm f$.
- (4) $g = \chi \pm \phi$ and $f = \phi$, where $\chi : S \rightarrow \mathbb{C}$ is a non-zero multiplicative function and $\phi : S \rightarrow \mathbb{C}$ is a solution of (12).

Theorem 1.2. *The solutions $g, f : S \rightarrow \mathbb{C}$ of the functional equation*

$$f(xy) = f(x)g(y) + f(y)g(x), \quad x, y \in S,$$

with $f \neq 0$ can be listed as follows:

- (1) $f = c(\chi_1 - \chi_2)$ and $g = \frac{\chi_1 + \chi_2}{2}$, where $\chi_1, \chi_2 : S \rightarrow \mathbb{C}$ are two different multiplicative functions and $c \in \mathbb{C} \setminus \{0\}$.
- (2) f is any non-zero function such that $f = 0$ on S^2 and $g = 0$.
- (3) $f = \phi$ and $g = \chi$, where $\chi : S \rightarrow \mathbb{C}$ is a non-zero multiplicative function and $\phi : S \rightarrow \mathbb{C}$ is a solution of (12).

Theorem 1.3. *The solutions $g, f : S \rightarrow \mathbb{C}$ of the functional equation*

$$g(x\sigma(y)) = g(x)f(y) - g(y)f(x), \quad x, y \in S,$$

with $g \neq 0$ are the following:

- (1) g is any non-zero function such that $g = 0$ on S^2 and $f = cg$, where $c \in \mathbb{C}$.
- (2) $g = c_1 \frac{\chi - \chi^*}{2}$ and $f = \frac{\chi + \chi^*}{2} + c_2 \frac{\chi - \chi^*}{2}$, where $\chi : S \rightarrow \mathbb{C}$ is a multiplicative function such that $\chi \neq \chi^*$ and $c_1 \in \mathbb{C} \setminus \{0\}$, $c_2 \in \mathbb{C}$ are constants.
- (3) $g = \phi$ and $f = \chi + c\phi$, where $\chi : S \rightarrow \mathbb{C}$ is a non-zero even multiplicative function, $\phi : S \rightarrow \mathbb{C}$ is an odd solution of (12) and $c \in \mathbb{C}$.

Remark 1.4. *The solution $\phi : S \rightarrow \mathbb{C}$ of (12) can be expressed in terms of multiplicative, additive, and arbitrary functions as follows (See Ebanks [2, Theorem 3.1 (B)]).*

$$\phi = \begin{cases} \chi^A & \text{on } S \setminus I_{\chi} \\ 0 & \text{on } I_{\chi} \setminus P_{\chi} \\ \rho & \text{on } P_{\chi} \end{cases},$$

where $\chi : S \rightarrow \mathbb{C}$ is a non-zero multiplicative function, $A : S \setminus I_{\chi} \rightarrow \mathbb{C}$ an additive function, and $\rho : P_{\chi} \rightarrow \mathbb{C}$ a function such that we have the following conditions.

- (I): If $x \in \{up, pv, upv\}$ for $p \in P_{\chi}$ and $u, v \in S \setminus I_{\chi}$, then we have respectively $\rho(x) = \rho(p)\chi(u)$, $\rho(x) = \rho(p)\chi(v)$, or $\rho(x) = \rho(p)\chi(uv)$.
- (II): $\phi(xy) = \phi(yx) = 0$ for all $x \in I_{\chi} \setminus P_{\chi}$ and $y \in S \setminus I_{\chi}$.

The following lemma will be used throughout the paper.

Lemma 1.5. *Let $f : S \rightarrow \mathbb{C}$ be a non-zero function satisfying*

$$f(x\sigma(y)) = \beta f(x)f(y), \quad \text{for all } x, y \in S, \tag{1}$$

where $\beta \in \mathbb{C} \setminus \{0\}$ is a constant. Then there exists a non-zero multiplicative function $\chi : S \rightarrow \mathbb{C}$ such that $\beta f = \chi$ and $\chi^* = \chi$.

Proof. By using the associativity of the semigroup operation we compute $f(x\sigma(y)\sigma(z))$ using the identity (1) first as $f((x\sigma(y))\sigma(z))$ and then as $f(x(\sigma(y)\sigma(z)))$ and compare the results to obtain

$$\beta^2 f(x)f(y)f(z) = \beta f(x)f(yz), \quad \text{for all } x, y, z \in S. \tag{2}$$

Since $f \neq 0$ and $\beta \neq 0$ Equation (2) can be written as

$$f(yz) = \beta f(y)f(z), \quad \text{for all } y, z \in S. \tag{3}$$

This implies that the function $\chi := \beta f$ is multiplicative. On the other hand

$$f(yz) = f(y\sigma(\sigma(z))) = \beta f(y)f^*(z), \quad \text{for all } y, z \in S. \tag{4}$$

Since $f \neq 0$ and $\beta \neq 0$, we get by comparing equation (3) and (4) that $f^* = f$, then $\chi^* = \chi$. This completes the proof of Lemma 1.5. \square

2. Introduction

Let S be a semigroup and $\sigma : S \rightarrow S$ an involutive automorphism. The cosine–sine functional equation is

$$f(x\sigma(y)) = f(x)g(y) + f(y)g(x) + h(x)h(y), \quad x, y \in S, \tag{5}$$

where $f, g : S \rightarrow \mathbb{C}$. This equation generalizes both the cosine subtraction formula

$$g(x\sigma(y)) = g(x)g(y) + f(x)f(y), \quad x, y \in S, \tag{6}$$

and the sine addition formula

$$f(x\sigma(y)) = f(x)g(y) + f(y)g(x), \quad x, y \in S. \tag{7}$$

This paper extends previous results about the functional equation

$$f(x\sigma(y)) = f(x)g(y) + f(y)g(x) - g(x)g(y), \quad x, y \in S, \tag{8}$$

for unknown functions $f, g : S \rightarrow \mathbb{C}$ on semigroups, where σ is an involutive automorphism. It should be mentioned that Equation (8) is a special case ($h = ig$) of Equation (5). In a previous paper, Ajebar and Elqorachi [4] gave the general solution of (8) on a semigroup generated by its squares. Stetkær [5] and Ebanks [2, 6] (with $\sigma = id$) obtained the solutions on semigroups. Here we find the general solution for all semigroups, and as an application we describe the solutions of the following functional equation

$$f(x\sigma(y)) = f(x)g(y) + f(y)g(x) + \alpha g(x\sigma(y)), \quad x, y \in S, \tag{9}$$

on semigroups, where $\alpha \in \mathbb{C} \setminus \{0\}$ is a fixed constant. Equation (9) is a generalization of the sine addition law (7). Stetkær [7, Theorem 3.1] solved the functional equation

$$f(xy) = f(x)f(y) - g(x)g(y) + \alpha g(xy), \quad x, y \in S,$$

on semigroups, and the solutions of the functional equation

$$f(x\sigma(y)) = f(x)g(y) - f(y)g(x) + \alpha g(x\sigma(y)), \quad x, y \in S, \tag{10}$$

were given by Zeglami et al. [8, Proposition 4.1] on topological groups. We extend these results to semigroups, and we relate (10) to the functional equation

$$f(x\sigma(y)) = f(x)g(y) - f(y)g(x), \quad x, y \in S,$$

which was solved by Ebanks [1, Corollary 4.3] on monoids, and by the authors [3, Proposition 3.2] on semigroups.

The functional equations (6) and (7) have been investigated by many authors, beginning with the case $S = (\mathbb{R}, +)$, $\sigma(x) = -x$. Equation (6) with $\sigma = id$ was solved on abelian groups by Vincze [9], and on general groups by Chung, Kannappan, and Ng [10]. The solutions of (6) and (7) were also described on topological groups by Poulsen and Stetkær [11]. Ajebar and Elqorachi [4, 12] described the solutions of (6) and (7) on semigroups generated by their squares. A more general description of the solutions of (7) with $\sigma = id$ was obtained recently by Ebanks [1, Theorem 2.1] and [2, Theorem 3.1] on a semigroup not necessarily generated by its squares. Ebanks [1, Theorem 4.1] obtained the solutions of (6) on monoids.

The cosine–sine functional equation (5) (with $\sigma = id$) was solved by Chung, Kannappan, and Ng [10] for the case that S is a group. Their results were extended by Ajebar and Elqorachi [4] to the case that S is a semigroup generated by its squares. The solutions of the special case $h = ig$, $\sigma = id$, namely

$$f(xy) = f(x)g(y) + f(y)g(x) - g(x)g(y), \quad x, y \in S, \tag{11}$$

are described by Stetkær [5] in terms of multiplicative functions and solutions of the sine addition formula

$$f(xy) = f(x)g(y) + f(y)g(x), \quad x, y \in S.$$

The purpose of the present paper is to solve the functional equations (6), (7), (8), and (10) on a semigroup S , where $\sigma : S \rightarrow S$ is an involutive automorphism. As an application of (8), we solve the functional equation (9).

3. Main results

The functional equations (6) and (7) have not yet been solved on general semigroups. The present paper accomplishes this. We derive explicit formulas for the solutions of (6), (7), (8), (9), and (10) on semigroups in terms of multiplicative functions and solutions $\phi : S \rightarrow \mathbb{C}$ of the special instance of the sine addition law

$$\phi(xy) = \phi(x)\chi(y) + \phi(y)\chi(x), \quad x, y \in S, \tag{12}$$

where $\chi : S \rightarrow \mathbb{C}$ is a multiplicative function.

3.1. Solutions of the cosine subtraction formula (6)

The most recent result on the cosine subtraction formula (6) on monoids is [2, Theorem 4.1]. In this section we solve (6) on general semigroups.

The next result gives the general solution of (6) on semigroups.

Theorem 3.1. *The solutions $f, g : S \rightarrow \mathbb{C}$ of the functional equation (6) can be listed as follows :*

- (1) $g = 0$ and $f = 0$.
- (2) g is any non-zero function such that $g = 0$ on S^2 , and $f = cg$, where $c \in \{i, -i\}$.
- (3) $g = \frac{1}{1 + \alpha^2}\chi$ and $f = \frac{\alpha}{1 + \alpha^2}\chi$, where $\alpha \in \mathbb{C} \setminus \{i, -i\}$ is a constant and $\chi : S \rightarrow \mathbb{C}$ is a non-zero multiplicative function such that $\chi^* = \chi$.
- (4) $g = \frac{\delta^{-1}\chi_1 + \delta\chi_2}{\delta^{-1} + \delta}$ and $f = \frac{\chi_2 - \chi_1}{\delta^{-1} + \delta}$, where $\delta \in \mathbb{C} \setminus \{0, i, -i\}$ and $\chi_1, \chi_2 : S \rightarrow \mathbb{C}$ are two different multiplicative functions such that $\chi_1^* = \chi_1$ and $\chi_2^* = \chi_2$.
- (5) $f = -i\phi$ and $g = \chi \pm \phi$, where $\chi : S \rightarrow \mathbb{C}$ is a non-zero even multiplicative function and $\phi : S \rightarrow \mathbb{C}$ is an even solution of (12).
- (6) $g = \frac{\chi + \chi^*}{2}$ and $f = \frac{\chi - \chi^*}{2i}$, where $\chi : S \rightarrow \mathbb{C}$ is a multiplicative function such that $\chi^* \neq \chi$.

Note that f and g are Abelian in each case.

Furthermore, if S is a topological semigroup and $g \in C(S)$, then

$f, \chi, \chi_1, \chi_2 \in C(S)$.

Proof. If $g = 0$, then $f = 0$. This is case (1). Hence from now on we assume that $g \neq 0$.

Suppose that $g = 0$ on S^2 . For all $x, y \in S$ we have $x\sigma(y) \in S^2$, so equation (6) gives

$$g(x)g(y) + f(x)f(y) = 0. \quad (13)$$

Since $g \neq 0$, equation (13) implies that $f = cg$ for some $c \in \mathbb{C}$. Substituting this into (13) yields

$$(c^2 + 1)g(x)g(y) = 0,$$

and since $g \neq 0$, we obtain $c^2 + 1 = 0$. Thus $c \in \{i, -i\}$. This case appears in part (2) of Theorem 3.1.

If $f = 0$, then equation (6) reduces to

$$g(x\sigma(y)) = g(x)g(y), \quad x, y \in S.$$

By Lemma 1.5, $g =: \chi$ is multiplicative and satisfies $\chi^* = \chi$. This corresponds to part (3) of Theorem 3.1 with $\alpha = 0$.

Now assume that $g \neq 0$ on S^2 , $f \neq 0$, and distinguish two cases depending on the linear dependence of f and g .

First case: f and g are linearly dependent.

There exists $\alpha \in \mathbb{C}$ such that $f = \alpha g$. Then equation (6) becomes

$$g(x\sigma(y)) = (1 + \alpha^2)g(x)g(y), \quad x, y \in S. \quad (14)$$

Since $g \neq 0$ on S^2 and $f \neq 0$, we have $\alpha \notin \{0, i, -i\}$. By Lemma 1.5,

$$\chi := (1 + \alpha^2)g$$

is multiplicative and satisfies $\chi^* = \chi$. This corresponds to case (3) of Theorem 3.1 with $\alpha \neq 0$.

Second case: f and g are linearly independent.

Using associativity, compute $g(x\sigma(y)\sigma(z))$ in two ways: first as $g((x\sigma(y))\sigma(z))$ and then as $g(x(\sigma(y)\sigma(z)))$. After comparing and rearranging, we obtain

$$f(x)[f(yz) - f(y)g(z)] + g(x)[g(yz) - g(y)g(z)] = f(z)f(x\sigma(y)). \quad (15)$$

Since $f \neq 0$, choose $z_0 \in S$ such that $f(z_0) \neq 0$ and define

$$h(y) = \frac{f(yz_0) - f(y)g(z_0)}{f(z_0)}, \quad k(y) = \frac{g(yz_0) - g(y)g(z_0)}{f(z_0)}.$$

Equation (15) becomes

$$f(x)h(y) + g(x)k(y) = f(x\sigma(y)). \quad (16)$$

Using (6) and $\sigma(\sigma(y)) = y$, the function k can be written as

$$k(y) = c_1g(y) + c_2f(y), \quad y \in S, \quad (17)$$

where

$$c_1 = \frac{g(\sigma(z_0)) - g(z_0)}{f(z_0)}, \quad c_2 = \frac{f(\sigma(z_0))}{f(z_0)}.$$

Substituting (16) into (15) yields

$$\begin{aligned} f(x)[f(yz) - f(y)g(z)] + g(x)[g(yz) - g(y)g(z)] \\ = f(x)f(z)h(y) + g(x)f(z)k(y). \end{aligned} \tag{18}$$

Since f and g are linearly independent, equation (18) implies

$$g(yz) = g(y)g(z) + f(z)k(y), \tag{19}$$

$$f(yz) = f(y)g(z) + f(z)h(y). \tag{20}$$

Substituting (17) into (19) gives

$$g(yz) = [g(z) + c_1f(z)]g(y) + c_2f(z)f(y). \tag{21}$$

On the other hand, applying (6) to $(y, \sigma(z))$ and using $\sigma(\sigma(z)) = z$ gives

$$g(yz) = g(y)g^*(z) + f(y)f^*(z). \tag{22}$$

Comparing (21) and (22), and using linear independence of f and g , we obtain

$$g^* = g + c_1f, \tag{23}$$

$$f^* = c_2f. \tag{24}$$

Since $f \neq 0$, equation (24) yields $c_2^2 = 1$. Thus either $f = f^*$ or $f = -f^*$.

Subcase A : If $f = f^*$, then in view of (23) we have

$$g = g^* + c_1f^* = g + c_1f + c_1f = g + 2c_1f.$$

This implies $2c_1f = 0$, and since $f \neq 0$, we obtain $c_1 = 0$. Hence $g^* = g$, and equation (6) can be written as

$$g(xy) = g(x)g(y) + f(x)f(y), \quad x, y \in S. \tag{25}$$

Defining $l := if$, equation (25) becomes

$$g(xy) = g(x)g(y) - l(x)l(y), \quad x, y \in S.$$

According to Theorem 1.1, and taking into account that f and g are linearly independent, we have the following possibilities:

(i)

$$g = \frac{\delta^{-1}\chi_1 + \delta\chi_2}{\delta^{-1} + \delta}, \quad l = \frac{\chi_1 - \chi_2}{i(\delta^{-1} + \delta)},$$

where $\delta \in \mathbb{C} \setminus \{0, i, -i\}$ is a constant and $\chi_1, \chi_2 : S \rightarrow \mathbb{C}$ are two multiplicative functions with $\chi_1 \neq \chi_2$. Since $g = g^*$, $f = f^*$, and $l = if$, we deduce

$$f = \frac{\chi_2 - \chi_1}{\delta^{-1} + \delta}, \quad \chi_1 = \chi_1^*, \quad \chi_2 = \chi_2^*.$$

This is case (4).

(ii)

$$g = \chi \pm l, \quad l = \phi,$$

where $\chi : S \rightarrow \mathbb{C}$ is a non-zero multiplicative function and $\phi : S \rightarrow \mathbb{C}$ is a solution of (12). Since $f^* = f$ and $g^* = g$, we have $\chi^* = \chi$ and $\phi^* = \phi$. In addition, $l = if$ implies $f = -i\phi$. This occurs in part (5).

Subcase B: $f^* = -f$. Subcase B.1: $c_1 = 0$. Then $g^* = g$, and equation (6) can be written as

$$g(xy) = g(x)g(y) - f(x)f(y), \quad x, y \in S.$$

Similarly to the previous case, by Theorem 1.1 and taking into account that f and g are linearly independent, we obtain two possibilities:

(i)

$$g = \frac{\delta^{-1}\chi_1 + \delta\chi_2}{\delta^{-1} + \delta}, \quad f = \frac{\chi_1 - \chi_2}{i(\delta^{-1} + \delta)},$$

where $\delta \in \mathbb{C} \setminus \{0, i, -i\}$ and $\chi_1, \chi_2 : S \rightarrow \mathbb{C}$ are different multiplicative functions. Since $f^* = -f$ and $g^* = g$, we have

$$\delta^{-1}(\chi_1 - \chi_1^*) + \delta(\chi_2 - \chi_2^*) = 0, \tag{26}$$

$$\chi_1 + \chi_1^* = \chi_2 + \chi_2^*. \quad (27)$$

By [13, Corollary 3.19], $\chi_1 = \chi_2^*$. Then (26) reduces to

$$(\delta^{-1} - \delta)(\chi_1 - \chi_1^*) = 0.$$

Since $\chi_1 \neq \chi_2$, we must have $\delta^{-1} - \delta = 0$, i.e., $\delta = \pm 1$. This is case (6) with $\chi_1 = \chi$ and $\chi_2 = \chi^*$.

(ii) $g = \chi \pm f$ and $f = \phi$, where $\chi : S \rightarrow \mathbb{C}$ is a non-zero multiplicative function and $\phi : S \rightarrow \mathbb{C}$ is a solution of (12).

If $g = \chi + f$, then $f^* = -f$ and $g^* = g$ imply $g = \chi^* - f$. Adding and subtracting this from $g = \chi + f$, we get

$$g = \frac{\chi + \chi^*}{2}, \quad f = \frac{\chi^* - \chi}{2}.$$

By assumption $f \neq 0$, so $\chi \neq \chi^*$. Substituting f and g into (6) leads to $\chi = \chi^*$, which is a contradiction.

If $g = \chi - f$, similarly we get

$$g = \frac{\chi + \chi^*}{2}, \quad f = \frac{\chi - \chi^*}{2},$$

which also leads to $\chi = \chi^*$ (so $f = 0$). This case does not occur.

Subcase B.2: $c_1 \neq 0$.

From (23), we have $g^*(xy) = g(xy) + c_1 f(xy)$ for all $x, y \in S$. Using (6) and (20), we obtain

$$\begin{aligned} g^*(x)g(y) + f^*(x)f(y) &= g(x)g^*(y) + f(x)f^*(y) \\ &\quad + c_1 [f(x)g(y) + f(y)h(x)]. \end{aligned}$$

Since $f^* = -f$, $g^* = g + c_1 f$, $c_1 \neq 0$, and $f \neq 0$, the identity reduces to $h = -g$. Then equation (20) becomes

$$f(yz) = f(y)g(z) - f(z)g(y), \quad y, z \in S. \quad (28)$$

Using associativity and (6), we get

$$\begin{aligned} f(x)g(yz) - g(x)f(yz) &= f(x)[g(y)g(z) + f(y)f(z)] \\ &\quad - g(x)[f(y)g(z) + f(z)g^*(y)]. \end{aligned}$$

By linear independence of f and g , we deduce

$$f(yz) = f(y)g(z) + f(z)g^*(y). \quad (29)$$

Comparing (28) and (29) and using linear independence, we get $g^* = -g$. Then (23) gives $-g = g + c_1 f$, so $2g + c_1 f = 0$, which contradicts the linear independence of f and g . Hence, this case does not occur.

For the converse, it is easily checked that forms (1)–(5) satisfy (6).

Finally, suppose S is a topological semigroup and $g \in C(S)$. In case (1) there is nothing to prove. If $f \neq 0$, the continuity of f follows from the continuity of g and equation (6): choose $y_0 \in S$ with $f(y_0) \neq 0$, then

$$f(x) = \frac{g(x\sigma(y_0)) - g(y_0)g(x)}{f(y_0)}, \quad x \in S.$$

The map $x \mapsto g(x\sigma(y_0))$ is continuous since $x \mapsto x\sigma(y_0)$ is continuous. In case (4), the continuity of χ_1 and χ_2 follows from [13, Theorem 3.18].

This completes the proof of Theorem 3.1. □

3.2. Solutions of the sine addition formula (7)

The solution of the sine subtraction formula

$$g(x\sigma(y)) = g(x)f(y) - g(y)f(x), \quad x, y \in S,$$

on a general monoid was given recently by Ebanks [2, Corollary 4.3]. Here we find the solutions of the sine addition formula (7) on semigroups.

Theorem 3.2. *The solutions $f, g : S \rightarrow \mathbb{C}$ of Equation (7) are the following pairs:*

- (1) $f = 0$ and g is arbitrary.
- (2) f is any non-zero function such that $f = 0$ on S^2 , while $g = 0$.
- (3) $f = \frac{1}{2\alpha}\chi$ and $g = \frac{1}{2}\chi$, where $\chi : S \rightarrow \mathbb{C}$ is a non-zero multiplicative function such that $\chi^* = \chi$ and $\alpha \in \mathbb{C} \setminus \{0\}$.

(4) $f = c(\chi_1 - \chi_2)$ and $g = \frac{\chi_1 + \chi_2}{2}$, where $\chi_1, \chi_2 : S \rightarrow \mathbb{C}$ are two different multiplicative functions such that $\chi_1^* = \chi_1$, $\chi_2^* = \chi_2$ and $c \in \mathbb{C} \setminus \{0\}$.

(5) $f = \phi$ and $g = \chi$ where $\chi : S \rightarrow \mathbb{C}$ is a non-zero even multiplicative function and $\phi : S \rightarrow \mathbb{C}$ is an even solution of (12).

Note that, off the exceptional case (1), f and g are Abelian.

Furthermore, off the exceptional case (1), if S is a topological semigroup and $f \in C(S)$, then $g, \chi, \chi_1, \chi_2 \in C(S)$.

Proof. If $f = 0$ then g will be arbitrary. This occurs in case (1). From now on we assume that $f \neq 0$. Suppose that $f = 0$ on S^2 . For all $x, y \in S$, we get from equation (7) that

$$f(x)g(y) + f(y)g(x) = 0, \tag{30}$$

since $f \neq 0$ we obtain from equation (30) that $g = cf$ where $c \in \mathbb{C}$ is a constant, then if we take this into account in equation (30) we get $2cf(x)f(y) = 0$, for all $x, y \in S$. This implies that $c = 0$ because $f \neq 0$. This occurs in part (2) of Theorem 3.2. Now we assume that $f \neq 0$ on S^2 and we discuss two cases according to whether f and g are linearly dependent or not.

First case : f and g are linearly dependent. There exists a constant $\alpha \in \mathbb{C}$ such that $g = \alpha f$, so equation (7) can be written as follows

$$f(x\sigma(y)) = 2\alpha f(x)f(y), \quad \text{for all } x, y \in S. \tag{31}$$

Since $f \neq 0$ on S^2 , then $\alpha \neq 0$. According to Lemma 1.5, the function $\chi := 2\alpha f$ is multiplicative and $\chi^* = \chi$. This is case (3).

Second case : f and g are linearly independent. By using the associativity of the semigroup operation we compute $f(x\sigma(y)\sigma(z))$ using equation (7) first as $f((x\sigma(y))\sigma(z))$ and then as $f(x(\sigma(y)\sigma(z)))$ and compare the results. We obtain after some rearrangement that

$$f(x)[g(yz) - g(y)g(z)] + g(x)[f(yz) - f(y)g(z)] = f(z)g(x\sigma(y)). \tag{32}$$

Since $f \neq 0$, there exists $z_0 \in S$ such that $f(z_0) \neq 0$ and hence

$$f(x)h(y) + g(x)k(y) = g(x\sigma(y)), \quad x, y \in S, \tag{33}$$

where

$$h(y) = \frac{g(yz_0) - g(y)g(z_0)}{f(z_0)},$$

and

$$k(y) = \frac{f(yz_0) - f(y)g(z_0)}{f(z_0)}.$$

By using (33), equation (32) becomes

$$\begin{aligned} f(x)[g(yz) - g(y)g(z)] + g(x)[f(yz) - f(y)g(z)] \\ = f(x)f(z)h(y) + g(x)f(z)k(y), \quad x, y, z \in S. \end{aligned} \tag{34}$$

Since f and g are linearly independent we deduce from (34) that

$$g(yz) = g(y)g(z) + f(z)h(y), \tag{35}$$

and

$$f(yz) = f(y)g(z) + f(z)k(y), \quad \text{for all } y, z \in S. \tag{36}$$

Equation (7) implies that $f(yz_0) = f(y)g^*(z_0) + f^*(z_0)g(y)$ for all $y \in S$, so

$$k(y) = \alpha f(y) + \beta g(y), \tag{37}$$

where

$$\alpha = \frac{g^*(z_0) - g(z_0)}{f(z_0)} \quad \text{and} \quad \beta = \frac{f^*(z_0)}{f(z_0)}.$$

By using (37), equation (36) can be written as follows

$$f(yz) = (g(z) + \alpha f(z))f(y) + \beta f(z)g(y), \quad y, z \in S.$$

On the other hand equation (7) implies that

$$f(yz) = f(y)g^*(z) + f^*(z)g(y), \quad y, z \in S.$$

By comparing these last two identities it follows from the linear independence of f and g that

$$g^* = g + \alpha f \tag{38}$$

$$f^* = \beta f. \tag{39}$$

Since $f \neq 0$ we get from (39) that $\beta \neq 0$ and $\beta^2 = 1$.

On the other hand for all $x, y \in S$ we have

$$\begin{aligned}
f(x\sigma(y)) &= \beta^{-1}f^*(x\sigma(y)) = \beta^{-1}f(\sigma(x)\sigma(\sigma(y))) \\
&= \beta^{-1}f^*(x)g^*(y) + \beta^{-1}f^*(y)g^*(x) \\
&= f(x)[g(y) + \alpha f(y)] + f(y)[g(x) + \alpha f(x)] \\
&= f(x)g(y) + f(y)g(x) + 2\alpha f(x)f(y) \\
&= f(x\sigma(y)) + 2\alpha f(x)f(y).
\end{aligned}$$

So $\alpha = 0$ since $f \neq 0$, and then $g^* = g$.

Subcase A: $\beta = -1$. In this case $f^* = -f$ and equation (7) becomes

$$f(xy) = f(x)g(y) - f(y)g(x), \quad x, y \in S. \quad (40)$$

By using the associativity of the semigroup operation and taking into account equation (35), we compute $f(xyz)$ using equation (40) first as $f(x(yz))$ and then as $f((xy)z)$ and compare the results. We obtain after some simplification that

$$2g(x)g(y) = -f(x)h(y) - f(y)h(x), \quad x, y \in S. \quad (41)$$

Since $f \neq 0$, equation (41) implies that $h = af + bg$ for some constants $a, b \in \mathbb{C}$, taking this into account in (41) we obtain

$$2g(x)g(y) = (-2af(y) - bg(y))f(x) - bg(x)f(y). \quad (42)$$

By using the linear independence of f and g , we deduce from (42) that

$$2g + bf = 0. \quad (43)$$

Equation (43) contradicts the fact that f and g are linearly independent. This case does not occur.

Subcase B: $\beta = 1$. Since $f^* = f$ and equation (7) becomes

$$f(xy) = f(x)g(y) + f(y)g(x). \quad (44)$$

Then according to Theorem 1.2 and taking into account that $f \neq 0$, $g \neq 0$, $f^* = f$ and $g^* = g$ we have the following possibilities :

(i) $f = c(\chi_1 - \chi_2)$ and $g = \frac{\chi_1 + \chi_2}{2}$, for some constant $c \in \mathbb{C} \setminus \{0\}$ and $\chi_1, \chi_2 : S \rightarrow \mathbb{C}$ are two multiplicative functions such that $\chi_1 \neq \chi_2$, $\chi_1^* = \chi_1$ and $\chi_2^* = \chi_2$. This is case (4).

(ii) $f = \phi$ and $g = \chi$ where $\chi : S \rightarrow \mathbb{C}$ is a non-zero multiplicative function and $\phi : S \rightarrow \mathbb{C}$ is a solution of (12). Since $f = f^*$ and $g = g^*$, we see that $\phi^* = \phi$ and $\chi^* = \chi$. This occurs in part (5) of Theorem 3.2.

Conversely we check by elementary computations that if f, g have one of the forms (1)–(5) then (f, g) is a solution of equation (7).

For the continuity statements, the continuity of g follows easily from the continuity of f and the functional equation (7). Let $y_0 \in S$ such that $f(y_0) \neq 0$, in view of (7) we have

$$g(x) = \frac{f(x\sigma(y_0)) - g(y_0)f(x)}{f(y_0)}, \quad x \in S.$$

The function $x \mapsto f(x\sigma(y_0))$ is continuous, since S is a topological semigroup so that the right translation $x \mapsto x\sigma(y_0)$ from S into S is continuous.

In case (4) we get the continuity of χ_1 and χ_2 by the help of [13, Theorem 3.18]. This completes the proof of Theorem 3.2. \square

3.3. Solutions of Equation (8) and Equation (9)

The next theorem gives the general solution of the functional equation (8). Like in Stetkær [5] and Ebanks [6] the solutions $\phi : S \rightarrow \mathbb{C}$ of (12) play an important role, and Stetkær [5, Theorem 3.3] is Theorem 3.3 with $\sigma = id$.

Theorem 3.3. *The solutions $f, g : S \rightarrow \mathbb{C}$ of the functional equation (8) can be listed as follows :*

(1) $f = 0$ and $g = 0$.

(2) f is any non-zero function such that $f = 0$ on S^2 , and $g = 0$.

(3) f is any non-zero function such that $f = 0$ on S^2 , and $g = 2f$.

(4) $f = \frac{\beta^2}{2\beta - 1}\chi$ and $g = \beta\chi$, where χ is a non-zero even multiplicative function and $\beta \in \mathbb{C} \setminus \left\{0, \frac{1}{2}\right\}$ is a constant.

(5) There exist two different, non-zero even multiplicative functions $\chi_1, \chi_2 : S \rightarrow \mathbb{C}$ and a constant $c_1 \in \mathbb{C} \setminus \{0, -1, 1\}$ such that

$$f = \frac{\chi_1 + \chi_2}{2} + \frac{c_1^2 + 1}{2c_1} \frac{\chi_1 - \chi_2}{2} \quad \text{and} \quad g = \frac{\chi_1 + \chi_2}{2} + c_1 \frac{\chi_1 - \chi_2}{2}.$$

(6) There exist a non-zero even multiplicative function $\chi : S \rightarrow \mathbb{C}$ and a non-zero even function $\phi : S \rightarrow \mathbb{C}$ satisfying (12) such that

$$\begin{cases} f = \frac{1}{2}\phi + \chi \\ g = \phi + \chi \end{cases}.$$

(7) There exist a non-zero even multiplicative function $\chi : S \rightarrow \mathbb{C}$ and a non-zero even function $\phi : S \rightarrow \mathbb{C}$ satisfying (12) such that

$$\begin{cases} f = \phi + \chi \\ g = \chi \end{cases}.$$

(8) There exist a non-zero multiplicative function $\chi : S \rightarrow \mathbb{C}$ such that $\chi \neq \chi^*$, $f = \frac{\chi + \chi^*}{2}$, and $g \in \{\chi, \chi^*\}$.

Note that f and g are Abelian in each case.

Furthermore, if S is a topological semigroup and $f, g \in C(S)$, then

$\chi, \phi, \chi_1, \chi_2, \chi^* \in C(S)$.

Proof. Let $f, g : S \rightarrow \mathbb{C}$ be a solution of Solutions of Equation (8). Define the function $\delta := i(f - g)$. For all $x, y \in S$ we have

$$\begin{aligned} f(x)f(y) + \delta(x)\delta(y) &= f(x)f(y) - (f(x) - g(x))(f(y) - g(y)) \\ &= f(x)f(y) - f(x)f(y) + f(x)g(y) + g(x)f(y) - g(x)g(y) \\ &= f(x)g(y) + f(y)g(x) - g(x)g(y). \end{aligned}$$

So the functional equation (8) can be written as

$$f(x\sigma(y)) = f(x)f(y) + \delta(x)\delta(y), \tag{45}$$

which means that the pair (f, δ) is a solution of Solutions of Equation (6). Now, the rest of the proof follows from Theorem 3.1. The converse and the topological statements are easy to verify. This completes the proof of Theorem 3.3. \square

In the following result, we give the general solution of the functional equation (9) on semigroups.

Corollary 3.4. *The solutions $f, g : S \rightarrow \mathbb{C}$ of equation (9) can be listed as follows :*

(1) $f = 0$ and $g = 0$.

(2) f is any non-zero function such that $f = 0$ on S^2 , and $g = 0$.

(3) g is any non-zero function such that $g = 0$ on S^2 , and $f = 0$.

(4) $f = \frac{\alpha\beta}{1-2\beta}\chi$ and $g = \beta\chi$, where χ is a non-zero even multiplicative function and $\beta \in \mathbb{C} \setminus \left\{0, \frac{1}{2}\right\}$ is a constant.

(5)

$$f = \frac{-\alpha}{2}(\chi_1 + \chi_2) + \frac{-\alpha}{2c_1}(\chi_1 - \chi_2) \text{ and } g = \frac{\chi_1 + \chi_2}{2} + c_1 \frac{\chi_1 - \chi_2}{2},$$

where $\chi_1, \chi_2 : S \rightarrow \mathbb{C}$ are two different, non-zero even multiplicative functions and $c_1 \in \mathbb{C} \setminus \{0, -1, 1\}$ is a constant.

(6) $f = -\alpha\chi$ and $g = \phi + \chi$ where $\chi : S \rightarrow \mathbb{C}$ is a non-zero even multiplicative function and $\phi : S \rightarrow \mathbb{C}$ is an even solution of (12).

(7) $f = -2\alpha\phi - \alpha\chi$ and $g = \chi$ where $\chi : S \rightarrow \mathbb{C}$ is a non-zero even multiplicative function and $\phi : S \rightarrow \mathbb{C}$ is an even solution of (12).

(8)

$$\begin{cases} f = -\alpha\chi \\ g = \chi^* \end{cases} \quad \text{or} \quad \begin{cases} f = -\alpha\chi^* \\ g = \chi \end{cases},$$

where $\chi : S \rightarrow \mathbb{C}$ is a non-zero multiplicative function such that $\chi^* \neq \chi$.

Note that f and g are Abelian in each case.

Furthermore, if S is a topological semigroup and $f, g \in C(S)$, then

$\chi, \chi_1, \chi_2, \chi^* \in C(S)$.

Proof. Let (f, g) be a solution of equation (9), we define the function

$F := \frac{1}{\alpha}f - g$. For all $x, y \in S$, we have

$$F(x\sigma(y)) = \frac{1}{\alpha}f(x\sigma(y)) - g(x\sigma(y)) = \frac{1}{\alpha}f(x)g(y) + \frac{1}{\alpha}f(y)g(x). \tag{46}$$

On the other hand, we have

$$F(x)g(y) + F(y)g(x) = \frac{1}{\alpha}f(x)g(y) + \frac{1}{\alpha}f(y)g(x) - 2g(x)g(y). \tag{47}$$

By taking into account equation (47), equation (46) becomes

$$F(x\sigma(y)) = F(x)g(y) + F(y)g(x) + 2g(x)g(y).$$

This implies that

$$\frac{-1}{2}F(x\sigma(y)) = \frac{-1}{2}F(x)g(y) + \frac{-1}{2}F(y)g(x) - g(x)g(y). \tag{48}$$

This means that $\left(\frac{-1}{2}F, g\right)$ satisfies equation (8). The rest of the proof follows from Theorem 3.3.

The converse and the continuity statements are easy to verify. This completes the proof. \square

3.4. Solutions of Equation (10)

In the following theorem we extend the results obtained in [8, Proposition 4.1] on topological groups to the case of semigroups.

Theorem 3.5. *The solutions $f, g : S \rightarrow \mathbb{C}$ of the functional equation (10) are the following pairs :*

- (1) $f = \alpha g$ and g is arbitrary.
- (2) g is any non-zero function such that $g = 0$ on S^2 , and $f = 0$.
- (3) f is any non-zero function such that $f = 0$ on S^2 , and $g = 0$.
- (4) f is any non-zero function such that $f = 0$ on S^2 , and $g = \frac{c}{\alpha(1+c)}f$, where $c \in \mathbb{C} \setminus \{0, -1\}$ is a constant.

(5)

$$f = \alpha \frac{\chi + \chi^*}{2} + \alpha(c_1 + c_2) \frac{\chi - \chi^*}{2} \quad \text{and} \quad g = \frac{\chi + \chi^*}{2} + c_2 \frac{\chi - \chi^*}{2},$$

where $\chi : S \rightarrow \mathbb{C}$ is a multiplicative function such that $\chi^* \neq \chi$ and $c_1 \in \mathbb{C} \setminus \{0\}$, $c_2 \in \mathbb{C}$ are constants.

- (6) $f = \alpha\chi + \alpha(c+1)\phi$ and $g = \chi + c\phi$, where $c \in \mathbb{C}$ is a constant, $\chi : S \rightarrow \mathbb{C}$ is a non-zero even multiplicative function and $\phi : S \rightarrow \mathbb{C}$ is an odd solution of (12).

Note that, off the exceptional case (1), f and g are Abelian.

Furthermore, if S is a topological semigroup and $f, g \in C(S)$, then

$$\chi, \chi^* \in C(S).$$

Proof. Let $f, g : S \rightarrow \mathbb{C}$ be a solution of the functional equation (10). If we put $F := \frac{1}{\alpha}f - g$, then for all $x, y \in S$ we have

$$F(x\sigma(y)) = \frac{1}{\alpha}f(x\sigma(y)) - g(x\sigma(y)) = \frac{1}{\alpha}f(x)g(y) - \frac{1}{\alpha}f(y)g(x). \quad (49)$$

On the other hand

$$F(x)g(y) - F(y)g(x) = \frac{1}{\alpha}f(x)g(y) - g(x)g(y) - \frac{1}{\alpha}f(y)g(x) + g(x)g(y). \quad (50)$$

Taking into account equation (50), equation (49) becomes

$$F(x\sigma(y)) = F(x)g(y) - F(y)g(x), \quad \text{for all } x, y \in S.$$

If $F = 0$, the g will be arbitrary and $f = \alpha g$. This is case (1). Now on we assume that $F \neq 0$. The rest of the proof follows easily from Theorem 1.3.

The converse and the topological statements are easily verified. This completes the proof of Theorem 3.5. \square

4. Applications

In this section, we will apply some of our results to two different types of semigroups.

Application 4.1. *Let $S = (\mathbb{R}, +)$ under the usual topology, and let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be the involution defined by $\sigma(x) = -x$ for all $x \in \mathbb{R}$. The functional equations (9), and (10) can be written respectively as follows*

$$f(x-y) = f(x)g(y) + f(y)g(x) + \alpha g(x-y), \quad (51)$$

$$f(x-y) = f(x)g(y) - f(y)g(x) + \alpha g(x-y), \quad (52)$$

where $f, g : \mathbb{R} \rightarrow \mathbb{C}$. We determine the continuous solutions of (51) and (52). The case $\alpha = 0$ of (52) is [13, Example 4.14], so the new feature here is the additional term $\alpha g(x-y)$. The continuous non-zero multiplicative functions on S are the functions

$$\chi(x) = e^{i\lambda x}, \quad x \in \mathbb{R}, \quad (53)$$

where $\lambda \in \mathbb{C}$. The only even, non-zero multiplicative function χ on S is $\chi = 1$. The only even, additive function a on S is $a = 0$.

According to Corollary 3.4, the solutions $f, g \in C(S)$ of (51) are the following:

- (a) $f = 0$ and $g = 0$.
- (b) $f = \frac{\alpha\beta}{1-2\beta}$ and $g = \beta$, where $\beta \in \mathbb{C} \setminus \left\{0, \frac{1}{2}\right\}$.
- (c) $f(x) = -\alpha e^{i\lambda x}$ and $g(x) = e^{-i\lambda x}$, where $\lambda \in \mathbb{C}$.
- (d) $f(x) = -\alpha e^{-i\lambda x}$ and $g(x) = e^{i\lambda x}$, where $\lambda \in \mathbb{C}$.

According to Theorem 3.5, the solutions $f, g \in C(S)$ of (52) can be listed as follows:

- (a) $f = \alpha g$ and $g \in C(S)$ arbitrary.
- (b) $f(x) = \frac{(1+c_1+c_2)\alpha}{2}e^{i\lambda x} + \frac{(1-c_1-c_2)\alpha}{2}e^{-i\lambda x}$ and $g(x) = \frac{1+c_2}{2}e^{i\lambda x} + \frac{1-c_2}{2}e^{-i\lambda x}$, where $\lambda \in \mathbb{C}$, $c_1 \in \mathbb{C} \setminus \{0\}$, and $c_2 \in \mathbb{C}$.

In the next application, we shall apply our theory to a non-abelian semigroup.

Application 4.2. Let $S = H_3$ be the Heisenberg group

$$H_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\},$$

and let

$$X = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

for all $x, y, z \in \mathbb{R}$. We consider the involution $\sigma : S \rightarrow S$ defined by

$$\sigma(X) = \begin{pmatrix} 1 & -x & z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}.$$

The continuous non-zero multiplicative functions on S have the form (See [13, Example 3.14])

$$\chi(X) = e^{ax+by},$$

where $a, b \in \mathbb{C}$. The only even, non-zero multiplicative function χ on S is $\chi = 1$. The only even, additive function on S is $a = 0$. The continuous solutions of equation (6) are the following three types:

- (a) $g = 0$ and $f = 0$.
- (b) $g = \frac{1}{1 + \alpha^2}$ and $f = \frac{\alpha}{1 + \alpha^2}$, where $\alpha \in \mathbb{C} \setminus \{i, -i\}$ is a constant.
- (c) $g(X) = \frac{e^{ax+by} + e^{-ax-by}}{2}$ and $f(X) = \frac{e^{ax+by} - e^{-ax-by}}{2i}$, where $a, b \in \mathbb{C}$.

The continuous solutions of equation (8) can be listed as follows:

- (a) $f = 0$ and $g = 0$.
- (b) $f = \frac{\beta^2}{2\beta - 1}$ and $g = \beta$, where $\beta \in \mathbb{C} \setminus \left\{0, \frac{1}{2}\right\}$ is a constant.
- (c) $f(X) = \frac{e^{ax+by} + e^{-ax-by}}{2}$ and $g(X) = e^{ax+by}$, where $a, b \in \mathbb{C}$.
- (d) $f(X) = \frac{e^{ax+by} + e^{-ax-by}}{2}$ and $g(X) = e^{-ax-by}$, where $a, b \in \mathbb{C}$.

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